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On the existence of the ex post symmetric random entry model[☆]

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ABSTRACT

This paper studies symmetry among countably infinitely many agents who randomly enter into a stochastic process, one for each period. Upon entry, they observe only the current period signal and try to draw inference about the underlying state governing the stochastic process. We show that there exist random entry models under which agents are ex post symmetric. That is, all agents have identical posterior belief about the underlying states, although they are not ex ante symmetric. The form of the posterior belief is uniquely pinned down by ex post symmetry and a stationarity condition. Our results provide a common prior foundation for the model studied in Liu and Skrzypacz (2014).

1. Introduction

This paper studies symmetry among countably infinitely many agents who randomly enter into a stochastic process, one for each period. Upon entry, they observe only the current period signal and try to draw inference about the underlying state governing the stochastic process. We investigate whether symmetry among these agents is consistent with common prior of entry.

It is well known that no random entry model can make infinitely many entering agents *ex ante symmetric*, i.e., that they have identical beliefs about when they enter prior to entry. Nonetheless, our first main result proves that there do exist random entry models that make infinitely many entering agents *ex post symmetric*, that is, they have identical posterior belief about the underlying state provided they have the same observation upon entry. The most important property of such entry models is that the length of entry can be unbounded but the whole process cannot last forever. Such ex post symmetric random entry models are not unique. But our second main result shows that if an additional stationarity condition is imposed, all these models are equivalent in the sense that they all result in the same form of posterior beliefs. This implies that in applications, the actual choice of such a random entry model is immaterial, as long as only posterior beliefs are concerned.[1]

We also show how our results can be applied to the reputation game analyzed in Liu and Skrzypacz (2014). In their model, an informed long-run player interacts with a sequence of uninformed short-lived players who enter the game at random times and only observe the long-run player's actions in the recent few periods. In order to focus on the symmetric behavior of the short-lived players, they assume that all short-lived players hold identical beliefs about when they enter, which is inconsistent with common prior of entry. Our existence result of ex post symmetric random entry models provides a remedy for this discrepancy. An easy application of our results shows that their model and analysis are indeed consistent with the common prior assumption, because the symmetric behavior of the short-lived players can be guaranteed if they are ex post symmetric.

The rest of the paper is organized as follows. Section 2 presents the random entry and learning model. We introduce the key notion of ex post symmetry. Section 3 contains the analysis and results of the paper. Section 4 provides an application and discuss how our results can be applied to the reputation game studied in Liu and Skrzypacz (2014). All missing proofs can be found in the Appendix.

2. Random entry and learning model

Let Ω be the set of all possible states of nature. For convenience, we assume that Ω is finite.[2] A generic element of Ω is denoted by ω. Let $\pi \in \Delta^{|\Omega|-1}$ be a prior distribution over the states with full support. The prior probability of a state $\omega \in \Omega$

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1 In a related paper, Boylan (1992) studied a pairwise random matching model among countably infinitely many agents, each of whom can be one of finitely many types. Although it cannot be the case that all matches are equally likely, he showed that as long as only matched types are concerned, there exist random matching models under which agents are symmetric in terms of their partners' types.

2 All the results in this paper also hold if Ω is countable.

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is denoted by π_ω. Let S be a finite signal space.[3] A generic signal will be denoted by $s \in S$. The set of the states of nature Ω and the signal space S together define an infinite dimensional measurable space $(\Omega \times S^\infty, \mathcal{G})$, where $\mathcal{G}$ is the usual product σ-algebra over $\Omega \times S^\infty$. Every probability measure P over $(\Omega \times S^\infty, \mathscr{G})$, whose marginal distribution over Ω is π, defines a signal process as follows. In period $t = 0$, nature selects a state $\omega \in \Omega$ according to the prior distribution π. In every period $t \geq 1$, a signal $s_t \in S$ is generated, and the evolution of the signals is governed by the marginal distribution of signals, given the realized state ω, $P_\omega(\cdot) \equiv P(\cdot \mid \{\omega\} \times S^\infty)$. Let $\mathcal{P}$ be the set of all such signal processes. To avoid trivial cases, we assume that both Ω and S contain at least two elements.

There are countably many agents, indexed by $i = 1, 2, \ldots$. The set of all agents is simply the set of natural numbers $\mathbb{N}$. Imagine the situation where agents enter into the signal process at random times, one agent for each period. Upon entry, each agent does not observe how long the signal process has been running. In other words, they do not know the calendar time of their entry. The only observation each agent has is just the current signal. Based on the observed signal, each agent then draws inference about the underlying state. Both the signal process $P \in \mathcal{P}$ and the random entry rule are common knowledge among the agents. We are interested in whether all the agents can draw identical inferences when they enter and observe the same signal. To answer this question, we first need to formalize their random entry and learning.

2.1. Random entry model

Intuitively, a random entry model specifies who enters in which period with what probability. Formally, for each $n = 1, 2, \ldots$, define

$$\Theta_n \equiv \left\{(i_1, i_2, \ldots, i_n) \in \mathbb{N}^n \mid i_s \neq i_t \text{ if } s \neq t\right\},$$

and define

$$\Theta_\infty \equiv \left\{(i_1, i_2, \ldots) \in \mathbb{N}^\infty \mid i_s \neq i_t \text{ if } s \neq t\right\}.$$

Let

$$\Theta \equiv \bigcup_{n=1}^{=\infty} \Theta_n$$

be the union of all these sets, where the upper bound "$= \infty$" means that $n = \infty$ is also included. Every element $\theta \in \Theta$ refers to a particular entry process. In particular, it specifies the length and order of this entry process. For example, the entry process $\theta = (i_1, i_2, \ldots, i_n) \in \Theta_n$ specifies that n agents, $i_1, i_2, \ldots, i_n$, will enter into the signal process. Agent i_1 enters in period $t = 1$. Agent i_2 enters in period $t = 2$, and so on. This entry process will continue until agent i_n enters in period $t = n$. For another example, an entry process $\theta \in \Theta_\infty$ lasts forever and it specifies the order of entry of infinitely many agents. The requirement that $i_s \neq i_t$ if $s \neq t$ in the above construction simply means that all agents are short-lived: each agent enters at most once. If an agent enters in period s, then he cannot enter in period t again. The set Θ contains all the possibilities of entry processes.

For every entry process $\theta \in \Theta$ and each agent $i \in \mathbb{N}$, define agent i's entry period as[4]

$$\tau_i(\theta) \equiv \begin{cases} t, & \text{if } i_t = i, \\ +\infty, & \text{if } i_s \neq i \text{ for all } s \geq 1. \end{cases} \quad (1)$$

Thus, the mapping $\tau_i : \Theta \to \mathbb{N} \cup \{+\infty\}$ is the random time at which agent i enters. For example, the set $(\tau_i = t) \equiv \{\theta \in \Theta \mid \tau_i(\theta) = t\}$ represents the event that agent i enters in period t and $(\tau_i = +\infty)$ is the event that agent i never enters. Let $\mathscr{E}$ be the smallest σ-algebra of Θ containing all the events of the form $(\tau_i = t)$ for some $i \geq 1$ and $t \geq 1$. We are now ready to define a random entry model.

Definition 1. A random entry model is a probability measure μ over $(\Theta, \mathscr{E})$.

The entry of agents governed by a random entry model μ can be understood as a two-stage randomization. In the first stage, the length $n = 1, 2, \ldots, +\infty$ of the entry process is realized according to the distributions $\{\mu(\Theta_n)\}_{n=1}^{=\infty}$. In the second stage, conditional on the realized length n, a particular entry process $\theta = (i_1, i_2, \ldots, i_n) \in \Theta_n$ or $\theta = (i_1, i_2, \ldots) \in \Theta_\infty$ if $n = \infty$ is chosen according to the conditional distribution $\mu(\cdot|\Theta_n)$. For finite n, agents $i_1, i_2, \ldots, i_n$ enter successively in each period $t = 1, \ldots, n$ and then the entry process ends. For $n = \infty$, agents $i_1, i_2, \ldots$ enter successively in each period and the entry process lasts forever.

A random entry model μ is also agents' common prior about the entry process. From this common prior, each agent forms his prior belief about when he enters. Denote by $\mu_i^t \equiv \mu(\tau_i = t)$ the probability that agent $i \in \mathbb{N}$ enters in period $t \in \mathbb{N}$.

Our formulation of a random entry model is very general. It treats many of the usual entry models as its special cases. A trivial example is that μ puts unit mass on the entry process $(1, 2, \ldots) \in \Theta_\infty$. In this entry model, agent i enters in period $t = i$ deterministically. Upon entry, agent i knows for sure that the current period is $t = i$ because $\mu_i^t = 1$ if $t = i$ and $\mu_i^t = 0$ if $t \neq i$. This entry model is widely used in the study of long-run interactions where one side is a sequence of short-lived players.[5] Another example is the symmetric entry of a finite population. That is, the random entry model μ is the uniform distribution over the set of all permutations of $1, \ldots, n$ for some finite n. In this case, all agents hold identical belief about their entry time. In particular, each agent's belief about his entry time is just the uniform distribution over $t = 1, \ldots, n$, i.e., $\mu_i^t = 1/n$ for $1 \leq i, t \leq n$. Thus, upon entry, every agent thinks that he is in one of the first n periods equally likely. This is the random entry model used in Guarino et al. (2011) and Monzón and Rapp (2014) for studying observational learning with position uncertainty of a finite population.

In both of these two examples, the length of entry is fixed. It is either infinity in the first example or a finite number of periods in the second example. However, our formulation of a random entry model has a much greater generality in terms of the entry length. In particular, it allows any arbitrary mixture between entry processes of different lengths. As we will see soon, this is crucial for our results.

2.2. Learning

Suppose the signal process is $P \in \mathcal{P}$ and the random entry model is μ. The two probability spaces $(\Omega \times R^\infty, \mathscr{G}, P)$ and $(\Theta, \mathscr{E}, \mu)$ together form a product probability space $(\Omega \times S^\infty \times \Theta, \mathscr{F}, P \otimes \mu)$, where $\mathscr{F}$ and $P \otimes \mu$ are the corresponding product σ-algebra and product measure respectively. The probability measure $P \otimes \mu$ now governs both the evolution of signals and entry. Notice that by this formulation, we implicitly assume that the signal process and entry are independent.

[3] Similarly, we assume finite signal space just for convenience. All the results can be extended to the case where S is a compact metric space, e.g., a real interval.

[4] By construction, for each $\theta \in \Theta$, there exists at most one t such that $i_t = i$.

[5] See, for example, Chapter 15 in Mailath and Samuelson (2006) for standard reputation games.

When agent $i \in \mathbb{N}$ enters, he only observes the current period signal s. Based on this information, he updates his belief about the underlying state $\omega \in \Omega$ from the common prior $P \otimes \mu$ via Bayes' rule. His posterior belief then becomes

$$\begin{aligned} \nu_i^{P\otimes\mu}(\omega|s) \equiv & P \otimes \mu \left(\omega \,\middle|\, s_{\tau_i} = s, \tau_i < \infty\right) \\ = & \frac{\pi_\omega P \otimes \mu \left(s_{\tau_i} = s,\ \tau_i < +\infty \,|\, \omega\right)}{\sum_{\omega'} \pi_{\omega'} P \otimes \mu \left(s_{\tau_i} = s,\ \tau_i < +\infty \,|\, \omega'\right)} \\ = & \frac{\pi_\omega \sum_{t=1}^{\infty} \mu_i^t P_\omega(s_t = s)}{\sum_{\omega'} \pi_{\omega'} \sum_{t=1}^{\infty} \mu_i^t P_{\omega'}(s_t = s)}, \end{aligned} \tag{2}$$

whenever the denominator is positive.

2.3. Ex post symmetry

From (2), it is clear that agents' posterior beliefs about the underlying states depend not only on the signal process, but also on their beliefs about their own entry time which is determined by the common random entry model μ. Generally speaking, two agents may draw different inferences about the states even if they have observed the same signal upon entry, because their beliefs about entry time differ. The objective of this paper is to seek those random entry models that lead to identical *posterior beliefs* across all entering agents *for all signal processes*. The following definition formalizes this idea.

Definition 2. A random entry model μ satisfies *ex post symmetry (EPS)* if for all $i, j \in \{k \in \mathbb{N} \,|\, \mu(\tau_k < \infty) > 0\}$ and every signal process $P \in \mathcal{P}$, we have

$$\nu_i^{P\otimes\mu}(\omega|s) = \nu_j^{P\otimes\mu}(\omega|s),\ \ \forall \omega \in \Omega,\ \ s \in S, \tag{3}$$

whenever both sides are well defined.

Hence, if a random entry model μ satisfies EPS, all entering agents are always *ex post symmetric*: their posterior beliefs about the underlying states are independent of their identities. Without any further requirement on a random entry model μ, the existence of μ that satisfies EPS is straightforward. In fact, any entry model that makes the entering agents *ex ante symmetric* satisfies EPS.

Definition 3. A random entry model μ satisfies *ex ante symmetry* if for all $i, j \in \{k \in \mathbb{N} \,|\, \mu(\tau_k < \infty) > 0\}$, we have

$$\mu_i^t = \mu_j^t,\ \ \forall t \geq 1.$$

All the entering agents under a random entry model that satisfies ex ante symmetry have identical belief about when they enter. According to (2), in all signal processes, these agents must have the same posterior belief about the underlying states, provided they observe the same signal. Thus, ex ante symmetry implies EPS. An example of such random entry models is the aforementioned uniform entry model of a finite population.

A more interesting question is whether there exists a random entry model that induces infinite population to enter and satisfies EPS. The answer to this question becomes much less obvious than the finite entry case. The difficulty is because of the following well-known impossibility result, which points out the conflict between entry of infinitely many agents and ex ante symmetry when agents have a common prior of the entry process.

Lemma 1. *If a random entry model μ induces an infinite population to enter, i.e., $|\{k \in \mathbb{N} \,|\, \mu(\tau_k < \infty) > 0\}| = \infty$, then the beliefs about entry time must be different across the entering agents. In other words, μ cannot satisfy ex ante symmetry.*

The reason behind Lemma 1 is rather simple. Suppose agent 1 enters in the first period with positive probability under a random entry model μ. If μ satisfies ex ante symmetry, then every agent who enters the process with positive probability must enter in the first period with the same probability as agent 1 does. If μ induces infinitely many agents to enter, the sum of the probabilities of each agent entering in the first period becomes infinity. This is impossible, since this sum is the probability of the event that someone enters in the first period.

In the next section, we show that there do exist random entry models that induce an infinite population to enter and satisfy EPS, despite the fact that the entering agents are *ex ante asymmetric*.

3. Results

3.1. A characterization of EPS

The following lemma provides a simple characterization of EPS. It reduces EPS into a condition on the prior belief ratios between every pair of entering agents. A random entry model μ satisfies EPS if and only if these ratios are constant over time for every pair of entering agents.

Lemma 2. *Let μ be a random entry model and $I \equiv \{i \in \mathbb{N} \,|\, \mu(\tau_i < \infty) > 0\}$ be the set of all entering agents. Then μ satisfies EPS if and only if for all $i, j \in I$, there exists $c_{ij} > 0$ such that*

$$\mu_i^t = c_{ij}\mu_j^t,\ \ \forall t \geq 1. \tag{4}$$

Lemma 2 provides a useful guidance for constructing the desired random entry models. It is straightforward that if (4) is satisfied for all pairs of entering agents, then μ satisfies EPS by the posterior belief formula (2). Conversely, condition (4) is a consequence of our requirement for EPS: the property that the entering agents with the same observation draw the same inference should hold not only for one particular signal process, but for all possible signal processes. In fact, if condition (4) is violated, then we can always construct a signal process in which the two agents have different posterior beliefs after observing some signal. Since $\mu_i^t/\mu_i^{t'}$ is the likelihood ratio of agent i entering in periods t and t', if $\mu_i^t/\mu_i^{t'} \neq \mu_j^t/\mu_j^{t'}$ for some entering agents i, j and some periods t and t', then these two agents have different conditional beliefs about when they enter, given that they enter in either period t or t'. Intuitively, in a signal process in which a signal only occurs in periods t and t', these two agents generally would have different posterior beliefs about the underlying states after observing this signal.[6]

Lemma 2 also implies that EPS imposes a restriction on the length of entry. Every random entry model that satisfies EPS must stop in finite time with probability one.

Corollary 1. *If the random entry model μ satisfies EPS, then $\mu(\Theta_\infty) = 0$.*

[6] For the necessity part to hold, we do require that the set of signal processes considered is rich enough. For example, if EPS is weakened to the requirement that (3) holds for only one particular signal process P, then condition (4) need not hold. One trivial example is a completely uninformative signal process P. For this signal process, (3) holds for every random entry model. However, the requirement that (3) holds for *every signal process* in the definition of EPS is stronger than necessary for condition (4). For condition (4), it is enough to have a sequence of signal processes $\{P^k\}_{k\geq 0}$ in which (i) $\{P_\omega^k\}_{k\geq 0}$ are identical for all but one state $\tilde{\omega}$; and (ii) in terms of the marginal distribution of signals in each period, $P_{\tilde{\omega}}^k$ differs from $P_{\tilde{\omega}}^0$ only in period $k \geq 1$. In a reputation model studied in Hu (2016), such sequence of signal processes can endogenously arise when the players adopt specific strategies (Lemmas A.8.2 and A.8.3).

The intuition behind this result is straightforward. If μ satisfies EPS, then the probability that some agents enter in period t is proportional to the probability that a certain agent, say agent 1, enters in period t. More importantly, the coefficient of proportionality is a constant over time. If the random entry model lasts forever with positive probability, i.e., $\mu(\Theta_\infty) > 0$, then the probability that some agent enters in period t must be bounded away from zero, for every t. This in turn requires that agent 1's probability of entering in period t be bounded away from zero, for every t. But we know this is impossible, since the total probability that agent 1 enters is bounded above by 1.

Notice that Corollary 1 does not imply that a random entry model that satisfies EPS must stop after a certain finite number of periods. But it does explain our formulation of random entry models. To have a random entry model that satisfies EPS and in which there is entry with positive probability in every period $t \geq 1$, it is necessary to allow mixture between entry processes of different lengths. This is the most important feature of our formulation of random entry models.

3.2. Main results

We now proceed to show that there do exist random entry models that satisfy EPS and in which there is entry with positive probability in every period $t \geq 1$. Corollary 1 implies that such random entry models must be a nontrivial mixture over the events $\{\Theta_n\}_{n\geq 1}$. The following condition imposes an additional requirement on the mixture.

Definition 4. A random entry model μ satisfies *stationarity (S)* if there exists $\delta \in (0, 1)$ such that

$$\mu\left(\bigcup_{k\geq n+1}\Theta_k\middle|\bigcup_{k\geq n}\Theta_k\right) = \delta,\ \forall n \geq 1.$$

In this case, we call δ the *continuation probability*.

Stationarity is best understood from an outside observer's point of view. This outside observer has the same common prior about the entry process as the agents. In addition, he observes each entry. If he observes that n agents have entered, then his belief about the next entry is $\mu(\cup_{k\geq n+1}\Theta_k|\cup_{k\geq n}\Theta_k)$. Thus, if μ satisfies S, the outside observer's belief will be stationary: he always believes that one new agent will enter in the next period with probability δ, regardless of how many agents have entered in the past.

It is easy to see that S is equivalent to the geometric distribution over the length of the entry process, i.e., $\mu(\Theta_n) = (1-\delta)\delta^{n-1}$ for all $n \geq 1$. In such a random entry model, infinitely many agents will enter. By Lemma 1, they must be ex ante asymmetric. Nonetheless, the following proposition, which is our first main result, shows that there exist random entry models that satisfy S and make all agents ex post symmetric.

Proposition 1. *For every $\delta \in (0, 1)$, there exists a random entry model that satisfies EPS and S with continuation probability δ.*

Proof. Fix any $\delta \in (0, 1)$. We explicitly construct a desired random entry model. For each $n \geq 1$, let $\mu(\{\theta_n\}) \equiv (1-\delta)\delta^{n-1}$ where $\theta_n = (n, n-1, \ldots, 1) \in \Theta_n$. For all other $\theta \in \Theta$, let $\mu(\{\theta\}) \equiv 0$. Clearly, μ is a probability measure over Θ. By construction, $\mu(\Theta_n) = \mu(\{\theta_n\}) = (1-\delta)\delta^{n-1}$. Hence, S is satisfied. Moreover, for each agent $i \geq 1$ and period $t \geq 1$, agent i enters in period t if and only if agent $i+t-1$ enters in period 1, if and only if θ_{i+t-1} is realized. Thus $\mu_i^t = \mu(\tau_i = t) = \mu(\{\theta_{i+t-1}\}) = (1-\delta)\delta^{i+t-1}$. Therefore, $\mu_i^t = \delta^{i-j}\mu_t^j$ for all agents pair $i, j \geq 1$ and period $t \geq 1$. By Lemma 2, μ satisfies EPS. □

The random entry model μ we constructed in the above proof is not the unique one that satisfies EPS and S. For instance, $\mu\circ\zeta^{-1}$ also satisfies all the required conditions, where $\zeta : \mathbb{N} \to \mathbb{N}$ is any permutation of the agents. Despite this multiplicity, the following proposition, which is our second main result, points out the "equivalence" between all these random entry models. That is, different random entry models that satisfy EPS and S with the same continuation probability must lead to the same posterior belief. This is because EPS and S, rather than the details of the random entry models, can jointly pin down the form of posterior beliefs.

Proposition 2. *Let μ be a random entry model that satisfies EPS and S with continuation probability $\delta \in (0, 1)$. If the signal process is $P \in \mathcal{P}$, then the common posterior belief among the entering agents is*

$$\nu^{P\otimes\mu}(\omega|s) = \frac{\pi_\omega \sum_{t=1}^\infty \delta^{t-1}P_\omega(s_t = s)}{\sum_{\omega'} \pi_{\omega'} \sum_{t=1}^\infty \delta^{t-1}P_{\omega'}(s_t = s)},\ \forall\omega \in \Omega, s \in S, \quad (5)$$

whenever the denominator is positive.

Proof. Without loss of generality, assume agent 1 enters with positive probability under μ. Because of EPS, we only need to show that agent 1's posterior belief is given by (5). By Lemma 2, for every agent $i \geq 1$, there exists $c_{i1} \geq 0$ such that $\mu_i^t = c_{i1}\mu_1^t$ for every period $t \geq 1$. Thus, $\sum_i \mu_i^t = \mu_1^t \sum_i c_{i1}$ for all $t \geq 1$. Because $\cup_i(\tau_i = t) = \cup_{n\geq t}\Theta_n$, we know

$$\sum_i \mu_i^t = \mu(\cup_i(\tau_i = t)) = \mu(\cup_{n\geq t}\Theta_n) = \sum_{n\geq t}(1-\delta)\delta^{n-1} = \delta^{t-1},$$

where the penultimate equality comes from S. Therefore, we have

$$\mu_1^t = \frac{\delta^{t-1}}{\sum_i c_{i1}},\ \forall t \geq 1.$$

Plugging this expression into (2) yields

$$\nu_1^P(\omega|s) = \frac{\pi_\omega \sum_{t=1}^\infty \delta^{t-1}P_\omega(s_t = s)}{\sum_{\omega'} \pi_{\omega'} \sum_{t=1}^\infty \delta^{t-1}P_{\omega'}(s_t = s)},\ \forall\omega \in \Omega,\ s \in S,$$

which is exactly (5). This completes the proof. □

4. Application

Liu and Skrzypacz (2014) study a reputation game between an informed long-run player and a sequence of uninformed short-lived players, where the short-lived players observe only the last few periods of the long-run player's actions instead of the full history. As is the case in standard reputation games, the long-run run player can be either a commitment type who plays the same action constantly over time, or a strategic type who chooses actions to maximize his discounted long-run expected payoff.

The short-lived players do not know the type of the long-run player. Upon entry, they have to form their posterior beliefs about the type of the long-run player, based on their priors and observations. Liu and Skrzypacz (2014) do not explicitly model the entry process of the short-lived players. Rather, to make the model tractable, Liu and Skrzypacz (2014) assume that all the agents have an identical prior about when they enter. One way to understand this assumption is to assume that the short-lived players in fact enter in a fixed order. Moreover, the short-lived agents are uncertain about their own identities and they hold an identical prior over their own identities.

Our model presents another view of this assumption by explicitly modeling the random entry process. In our entry model, the short-lived agents know their own identities, but are uncertain about their entering period. On the one hand, Lemma 1 shows that the assumption of common prior over entering time is not consistent with any entry model with infinitely many agents. On the other hand, our main results, Propositions 1 and 2, also provide an easy remedy to reconcile this discrepancy.

To apply our results, the major change we need to make to Liu and Skrzypacz (2014) is simply a reinterpretation of the long-run player's discount factor δ. To see this, let $\Omega \equiv \{\widetilde{\xi}, \widehat{\xi}\}$ be the type space of the long-run player. Type $\widetilde{\xi}$ represents the strategic type, while $\widehat{\xi}$ is the commitment type. The prior belief over the types is denoted by π. Let μ be a random entry model that satisfies EPS and S with continuation probability $\delta \in (0, 1)$. Its existence is guaranteed by our Proposition 1. Both π and μ are common knowledge among all the players. The timing of the game is as follows. In period $t = 0$, nature selects the type of the long-run player according to π and an entry process $\theta \in \Theta$ according to μ. From period $t = 1$ on, the short-lived players enter according to θ until the entry process ends. Upon entry, the entering agent observes the recent $K \geq 1$ periods of the long-run player's actions. If the entering agent happens to enter in period $1 \leq k \leq K - 1$, then he observes the full history. Then the long-run player and the short-lived player simultaneously choose an action $a_1 \in A_1$ and $a_2 \in A_2$ respectively.

On the short-lived players' side, let $S \equiv \cup_{k=0}^{K} A^k$ be the signal space.[7] The strategies of the strategic and commitment types of the long-run player induce probability distributions, $P_{\widetilde{\xi}}$ and $P_{\widehat{\xi}}$ respectively, over S^∞. The prior belief π together with $P_{\widetilde{\xi}}$ and $P_{\widehat{x}}$ then form a signal process P over $\Omega \times S^\infty$. Because μ satisfies EPS, we know that given any strategy of the strategic type, all the entering short-lived players will have the same posterior belief about the type of the long-run player if they observe the same signal, i.e., the recent K periods of history, upon entry. If, in addition, the strategic type plays a stationary strategy, i.e., a strategy that depends only on the public history $s \in S$, then all the entering short-lived players will have the same expectation about the long-run player's behavior given the same observation. This allows us to restrict attention to the symmetric strategies of the short-lived players.

On the long-run player's side, we assume that the strategic type does not discount the future payoffs. However, he does take into account the possibility that no one will enter from tomorrow on. Because μ is common knowledge and it satisfies S, the strategic type will always believe that the game will continue with probability δ regardless of how many short-lived players have entered. Thus, it is well known that the strategic type's incentive will be exactly the same as that when he discounts the future payoffs with discount factor δ but believes that the relationship will last forever.

Therefore, the model we proposed here and the one studied in Liu and Skrzypacz (2014) will have the same set of stationary equilibria. Moreover, our Proposition 2 shows the choice of the entry model is immaterial because all random entry models that satisfy EPS and S with the same continuation probability will lead to an identical posterior belief. Hence, our results provide a solid mathematical foundation for the model studied in Liu and Skrzypacz (2014).

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

[7] Here, A^k for $0 \leq k \leq K-1$ represents all possible observations in the first $K-1$ periods.

Appendix. Missing proofs

Proof of Lemma 1. Because a random entry model is mathematically equivalent to a random (partial) matching scheme between the agents and calendar times, this lemma is essentially a well-known impossibility result in random matching between an infinite number of agents. See, for example, Section 3 in Boylan (1992). We include its proof here for completeness.

Suppose by contradiction such a random entry model μ exists. Let $I \equiv \{i \in \mathbb{N} \mid \mu(\tau_i < \infty) > 0\}$ be the set of agents who enter with positive probability. By assumption, I is infinite. Pick any arbitrary $i \in I$. Pick any period $t \in \mathbb{N}$ such that $\mu_i^t > 0$. Because $\mu_j^t = \mu_i^t$ for all $j \in I$, we have $1 \geq \mu(\cup_{j\in I}(\tau_j = t)) = \sum_{j\in I} \mu_t^j = \sum_{j\in I} \mu_i^t = +\infty$, a contradiction. □

Proof of Lemma 2. Sufficiency is straightforward from the posterior belief formula (2). We only show necessity. Suppose μ satisfies EPS and $i, j \in I$. Pick any $\omega \in \Omega$ and $s \in S$. Pick a signal process $P \in \mathcal{P}$ such that both

$$\sum_{\omega'\neq\omega} \pi_{\omega'} \sum_{t=1}^{\infty} \mu_i^t P_{\omega'}(s_t = s) > 0 \text{ and } \sum_{\omega'\neq\omega} \pi_{\omega'} \sum_{t=1}^{\infty} \mu_j^t P_{\omega'}(s_t = s) > 0$$

hold. Because $i, j \in I$, such P exists. Define

$$c_{ij} = \frac{\sum_{\omega'\neq\omega} \pi_{\omega'} \sum_{t=1}^{\infty} \mu_i^t P_{\omega'}(s_t = s)}{\sum_{\omega'\neq\omega} \pi_{\omega'} \sum_{t=1}^{\infty} \mu_j^t P_{\omega'}(s_t = s)} > 0.$$

For each $k \geq 1$, find a signal process $P^k \in \mathcal{P}$ such that (i) $P_{\omega'}^k = P_{\omega'}$ for all $\omega' \neq \omega$, and (ii) $P_\omega^k(s_t = s) > 0$ if and only if $t = k$. EPS then implies $\nu_i^{P^k\otimes\mu}(\omega|s) = \nu_j^{P^k\otimes\mu}(\omega|s)$ for all $k \geq 1$. Equivalently, for each $k \geq 1$, we have

$$\sum_{t=1}^{\infty}(\mu_i^t - c_{ij}\mu_j^t)P_\omega^k(s_t = s) = 0.$$

By our construction of P^k, the above equation boils down to

$$(\mu_i^k - c_{ij}\mu_j^k)P^k(s_k = s) = 0,$$

implying $\mu_i^k = c_{ij}\mu_j^k$. Since $k \geq 1$ is arbitrary, this completes the proof. □

Proof of Corollary 1. Suppose μ satisfies EPS. Without loss of generality, assume agent 1 enters with positive probability, i.e., $\mu(\tau_1 < \infty) > 0$. By Lemma 2, for each agent $i \in \mathbb{N}$, there exists $c_{i1} \geq 0$ such that $\mu_i^t = c_{i1}\mu_1^t$ for all $t \geq 1$.[8] For each $t \geq 1$, because $\Theta_\infty \subset \cup_i(\tau_i = t)$, we know $\sum_i \mu_i^t = \mu(\cup_i(\tau_i = t)) \geq \mu(\Theta_\infty)$. This implies $(\sum_i c_{i1})\mu_1^t \geq \mu(\Theta_\infty)$ for all $t \geq 1$, or equivalently

$$\mu_1^t \geq \frac{\mu(\Theta_\infty)}{\sum_i c_{i1}}, \quad \forall t \geq 1.$$

Then we have

$$1 \geq \mu(\tau_1 < \infty) = \sum_t \mu_1^t \geq \sum_t \frac{\mu(\Theta_\infty)}{\sum_i c_{i1}}.$$

For this inequality to hold, we must have $\mu(\Theta_\infty) = 0$. □

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[8] If agent i never enters, $c_{i1} = 0$.

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