Online Appendix for "Optimal Contingent Delegation"

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This online appendix contains missing proofs. Section A provides the missing proof of Lemma 12. Section B provides the proof of Theorem 3 in Appendix D.1. Section C contains the proofs for Section 4.

Online Appendix A Missing Proof of Lemma 12

In Appendix B.3, we have proved Lemma 12 assuming that there exist desired h_1 and h_2 that satisfy parts (i) and (ii) of Lemma 12. The next lemma confirms the existence of such h_1 and h_2.

Lemma A.1. *For every $s_1 \in [\underline{L}_1, \bar{H}_1]$, there exists a unique $h_2(s_1) \in [c_2^*(s_1), d_2^*(s_1)]$ such that the following equation holds*

$$s_1 = \frac{h_2(s_1) - c_2^*(s_1)}{d_2^*(s_1) - c_2^*(s_1)} d_1^*(h_2(s_1)) + \frac{d_2^*(s_1) - h_2(s_1)}{d_2^*(s_1) - c_2^*(s_1)} c_1^*(h_2(s_1)). \tag{A.1}$$

Then, $h_1 \equiv h_2^{-1}$ and h_2 satisfy parts (i) and (ii) of Lemma 12.

Proof. For every $s_1 \in [\underline{L}_1, \bar{H}_1]$ and $s_2 \in [c_2^*(s_1), d_2^*(s_1)]$, define

$$g(s_1, s_2) \equiv \frac{s_2 - c_2^*(s_1)}{d_2^*(s_1) - c_2^*(s_1)} d_1^*(s_2) + \frac{d_2^*(s_1) - s_2}{d_2^*(s_1) - c_2^*(s_1)} c_1^*(s_2). \tag{A.2}$$

It is well defined by condition U and continuous by Lemma 2. We divide the remaining proof into several small steps.

Step 1: For every s_1, $g(s_1, \cdot)$ is strictly increasing.

Consider $c_2^*(s_1) \le s_2 < s_2' \le d_2^*(s_1)$. We have

$$\begin{aligned} g(s_1, s_2) &\le \frac{s_2 - c_2^*(s_1)}{d_2^*(s_1) - c_2^*(s_1)} d_1^*(s_2') + \frac{d_2^*(s_1) - s_2}{d_2^*(s_1) - c_2^*(s_1)} c_1^*(s_2') \\ &= \frac{s_2 - c_2^*(s_1)}{d_2^*(s_1) - c_2^*(s_1)} (d_1^*(s_2') - c_1^*(s_2')) + c_1^*(s_2') \\ &< \frac{s_2' - c_2^*(s_1)}{d_2^*(s_1) - c_2^*(s_1)} (d_1^*(s_2') - c_1^*(s_2')) + c_1^*(s_2') \\ &= g(s_1, s_2'), \end{aligned}$$

where the first inequality comes from monotonicity of c_1^* and d_1^* by Lemma 2. The second inequality comes from $d_1^*(s_2') > c_1^*(s_2')$ by condition U.

Step 2: If $s_1 = \underline{L}_1$, the unique $h_2(s_1) \in [c_2^*(\underline{L}_1),\ d_2^*(\underline{L}_1)]$ that satisfies $g(s_1, h_2(s_1)) = s_1$ is $h_2(s_1) = \underline{L}_2$.

Because $c_2^*(\underline{L}_1) = \underline{L}_2$ and $c_1^*(\underline{L}_2) = \underline{L}_1$, it is straightforward to see $g(\underline{L}_1, \underline{L}_2) = \underline{L}_1$. Uniqueness comes from the previous step.

Step 3: If $s_1 = \bar{H}_1$, the unique $h_2(s_1) \in [c_2^*(\bar{H}_1),\ d_2^*(\bar{H}_1)]$ that satisfies $g(s_1, h_2(s_1)) = s_1$ is $h_2(s_1) = \bar{H}_2$.

The proof is similar to the previous one.

Step 4: If $s_1 \in (\underline{L}_1,\ \bar{H}_1)$, then there exists a unique $h_2(s_1) \in (c_2^*(s_1),\ d_2^*(s_1))$ such that $g(s_1, h_2(s_1)) = s_1$.

It is easy to see $g(s_1, c_2^*(s_1)) = c_1^*(c_2^*(s_1))$. Because $s_1 > \underline{L}_1$, we then know $g(s_1, c_2^*(s_1)) < s_1$ by Lemma 9. Similarly, because $g(s_1, d_2^*(s_1)) = d_1^*(d_2^*(s_1))$ and $s_1 < \bar{H}_1$, we know $g(s_1, d_2^*(s_1)) > s_1$ by Lemma 9 again. Thus, by Step 1, we know there exists a unique $h_2(s_1) \in (c_2^*(s_1),\ d_2^*(s_1))$ such that $g(s_1, h_2(s_1)) = s_1$.

Step 5: $h_2 : [\underline{L}_1,\ \bar{H}_1] \to [\underline{L}_2,\ \bar{H}_2]$ is continuous and surjective.

Let $\{s_1^n\}_{n\geq 1} \subset [\underline{L}_1, \bar{H}_1]$ be a sequence converging to $s_1 \in [\underline{L}_1, \bar{H}_1]$. Because $\{h_2(s_1^n)\}_{n\geq 1} \subset [\underline{L}_2, \bar{H}_2]$, it has a convergent subsequence $\{h_2(s_1^{n_k})\}_{k\geq 1}$. Let $s_2 \equiv \lim_{k\to\infty} h_2(s_1^{n_k}) \in [c_2^*(s_1),\ d_2^*(s_1)]$. Because $g(s_1^{n_k}, h_2(s_1^{n_k})) = s_1^{n_k}$ for all $k \geq 1$ and g is continuous, we know $g(s_1, s_2) = s_1$. By Steps 2 - 4, we know $s_2 = h_2(s_1)$. This proves the continuity of h_2. Because $h_2(\underline{L}_1) = \underline{L}_2$ and $h_2(\bar{H}_1) = \bar{H}_2$ by Steps 2 and 3, we know h_2 is surjective since it is continuous.

Step 6: $h_2(\underline{L}_1) < h_2(s_1) < h_2(\bar{H}_1)$ for all $s_1 \in (\underline{L}_1, \bar{H}_1)$.

For all $s_1 \in (\underline{L}_1, \bar{H}_1)$, we have

$$h_2(\underline{L}_1) = \underline{L}_2 = c_2^*(\underline{L}_1) \leq c_2^*(s_1) < h_2(s_1) < d_2^*(s_1) \leq d_2^*(\bar{H}_1) = \bar{H}_2 = h_2(\bar{H}_1),$$

where the first and last equalities come from Steps 2 and 3. The two weak inequalities come from monotonicity of c_2^* and d_2^*. The two strict inequalities come from Step 4.

Step 7: $h_2 : [\underline{L}_1, \bar{H}_1] \to [\underline{L}_2, \bar{H}_2]$ is strictly increasing.

We first argue that h_2 is injective. Consider $\underline{L}_1 \leq s_1 < s_1' \leq \bar{H}_1$. Suppose, by contradiction, $h_2(s_1) = h_2(s_1') \equiv s_2$. By Step 6, we know $\underline{L}_1 < s_1 < s_1' < \bar{H}_1$. Thus, $c_2^*(s_1) < s_2 < d_2^*(s_1)$ and $c_2^*(s_1') < s_2 < d_2^*(s_1')$ by Step 4.

Because $g(s_1, s_2) = s_1 < s_1' = g(s_1', s_2)$ and $d_1^*(s_2) > c_1^*(s_2)$, we can directly see from (A.2) that

$$\frac{s_2 - c_2^*(s_1)}{d_2^*(s_1) - c_2^*(s_1)} < \frac{s_2 - c_2^*(s_1')}{d_2^*(s_1') - c_2^*(s_1')},$$

which implies

$$\frac{d_2^*(s_1) - s_2}{s_2 - c_2^*(s_1)} > \frac{d_2^*(s_1') - s_2}{s_2 - c_2^*(s_1')}.$$

But this is impossible, since $0 < s_2 - c_2^*(s_1') \le s_2 - c_2^*(s_1')$ and $0 < d_2^*(s_1) - s_2 \le d_2^*(s_1') - s_2$. Therefore, h_2 is injective.

Because h_2 is continuous by Step 5, we now know h_2 is strictly monotone. Because $h_2(\underline{L}_1) < h_2(\bar{H}_1)$, we know h_2 is strictly increasing.

The above Steps 2 - 4 and 7 together guarantee that h_2 satisfies parts (i) and (ii) in Lemma 12. These steps, together with Step 5, guarantee that $h_1 \equiv h_2^{-1} : [\underline{L}_2, \bar{H}_2] \to [\underline{L}_1, \bar{H}_1]$ is well defined and satisfies part (i).

Step 8: For all $s_2 \in (\underline{L}_2, \bar{H}_2)$, $h_1(s_1) \in (c_1^*(s_2), d_1^*(s_2))$. That is, h_1 satisfies part (ii).

Let $s_1 \equiv h_1(s_2) \in (\underline{L}_1, \bar{H}_1)$. Then, (A.1) can be written as

$$h_1(s_2) = \frac{h_2(s_1) - c_2^*(s_1)}{d_2^*(s_1) - c_2^*(s_1)} d_1^*(s_2) + \frac{d_2^*(s_1) - h_2(s_1)}{d_2^*(s_1) - c_2^*(s_1)} c_1^*(s_2).$$

Because $\frac{h_2(s_1) - c_2^*(s_1)}{d_2^*(s_1) - c_2^*(s_1)} \in (0, 1)$ by Step 4, we immediately know $h_1(s_2) \in (c_1^*(s_2), d_1^*(s_2))$. This completes the proof. □

Online Appendix B Proof of Theorem 3

Proof of Theorem 3. For notational simplicity, we write $a_i^*(s_i, s_{-i})$ for $\sigma_i^\phi(s_i, s_{-i})$. The goal is to show that $a^* \equiv (a_1^*, a_2^*)$ solves the following problem, which is equivalent to (1) by the standard envelope theorem argument:

$$\max_{(a_1, a_2)} \iint \Big(u_0(a_1(s_1, s_2), a_2(s_1, s_2)) + \sum_i u_i(a_i(s_i, s_{-i}), s_i) \Big) f_1(s_1) f_2(s_2)\, \mathrm{d}s_1 \mathrm{d}s_2, \tag{B.1}$$

subject to:

$$s_i a_i(s_i, s_{-i}) - \frac{a_i(s_i, s_{-i})^2}{2} = \int_0^{s_i} a_i(\tilde{s}_i, s_{-i}) \mathrm{d}\tilde{s}_i - \frac{a_i(0, s_{-i})^2}{2}, \ \forall i, s_i, s_{-i},$$

$a_i(s_i, s_{-i})$ is increasing in s_i, $\forall i, s_{-i}$.

Define the following (cumulative) Lagrange multiplier:

$$\Lambda_i(s_i, s_{-i}) = \begin{cases} f_{-i}(s_{-i})(1 - \kappa_i F_i(s_i)), & s_i \in [0,\, \underline{\phi}_i(s_{-i})], \\ f_{-i}(s_{-i})(1 - \frac{\partial w_i}{\partial a_i}(s_i, s_i, s_{-i}) f_i(s_i)), & s_i \in (\underline{\phi}_i(s_{-i}),\, \bar{\phi}_i(s_{-i})), \\ f_{-i}(s_{-i})(1 + \kappa_i(1 - F_i(s_i))), & s_i \in [\bar{\phi}_i(s_{-i}),\, 1]. \end{cases}$$

We argue that, for every s_{-i}, the following function is increasing in s_i:

$$\begin{aligned} &\Lambda_i(s_i, s_{-i}) + \kappa_i f_{-i}(s_{-i}) F_i(s_i) \\ =& \begin{cases} f_{-i}(s_{-i}), & s_i \in [0,\, \underline{\phi}_i(s_{-i})], \\ f_{-i}(s_{-i})(1 + \kappa_i F_i(s_i) - \frac{\partial w_i}{\partial a_i}(s_i, s_i, s_{-i}) f_i(s_i)), & s_i \in (\underline{\phi}_i(s_{-i}),\, \bar{\phi}_i(s_{-i})), \\ f_{-i}(s_{-i})(1 + \kappa_i), & s_i \in [\bar{\phi}_i(s_{-i}),\, 1], \end{cases} \end{aligned}$$

Clearly, it is increasing over $[0,\, \underline{\phi}_i(s_{-i})]$ and $[\bar{\phi}_i(s_{-i}),\, 1]$. By condition C1, it is also increasing over $[\underline{\phi}_i(s_{-i}),\, \bar{\phi}_i(s_{-i})]$. Hence, to show that it is increasing over $[0, 1]$, it suffices to verify the following two inequalities:

$$\kappa_i F_i(\underline{\phi}_i(s_{-i})) \geq \frac{\partial w_i}{\partial a_i}(\underline{\phi}_i(s_{-i}), \underline{\phi}_i(s_{-i}), s_{-i}) f_i(\underline{\phi}_i(s_{-i})), \tag{B.2}$$

$$\kappa_i(1 - F_i(\bar{\phi}_i(s_{-i}))) \geq -\frac{\partial w_i}{\partial a_i}(\bar{\phi}_i(s_{-i}), \bar{\phi}_i(s_{-i}), s_{-i}) f_i(\bar{\phi}_i(s_{-i})). \tag{B.3}$$

If $\underline{\phi}_i(s_{-i}) = 0$, (B.2) is directly implied by condition C2′. If $\underline{\phi}_i(s_{-i}) > 0$, we know from condition C2 that

$$g(s_i) = (s_i - \underline{\phi}_i(s_{-i}))\kappa_i F_i(s_i) - \int_0^{s_i} \frac{\partial w_i}{\partial a_i}(\underline{\phi}_i(s_{-i}), \tilde{s}_i, s_{-i}) f_i(\tilde{s}_i) \mathrm{d}\tilde{s}_i \leq 0, \;\; \forall s_i \in [0, \underline{\phi}_i(s_{-i})],$$

with equality at $\underline{\phi}_i(s_{-i})$. This implies that $g'(\underline{\phi}_i(s_{-i})) \geq 0$. Equivalently, (B.2) holds. Using conditions C3 and C3′, we can similarly verify that (B.3) also holds.

For every s_{-i}, being the difference of two increasing functions, $\Lambda_i(s_i, s_{-i})$ as a function of s_i has bounded variation. As a result, it induces a well-defined (signed) measure $\Lambda_i(\mathrm{d}s_i, s_{-i})$ over $[0, 1]$. Let

$$\Phi \equiv \{\text{direct mechanism } (a_1, a_2) \,|\, a_i(s_i, s_{-i}) \text{ is increasing in } s_i\}.$$

Define the Lagrangian function $\mathcal{L} : \Phi \to \mathbb{R}$ as, for every $a \in \Phi$,

$$\begin{aligned} \mathcal{L}(a) \equiv& \iint \Big(u_0(a_1(s_1, s_2), a_2(s_1, s_2)) + \sum_i u_i(a_i(s_i, s_{-i}), s_i)\Big) f_1(s_1) f_2(s_2)\, \mathrm{d}s_1 \mathrm{d}s_2 \\ &- \sum_i \iint \Big(\int_0^{s_i} a_i(\tilde{s}_i, s_{-i}) \mathrm{d}\tilde{s}_i - \frac{a_i(0, s_{-i})^2}{2} - s_i a_i(s_i, s_{-i}) + \frac{a_i(s_i, s_{-i})^2}{2}\Big) \Lambda_i(\mathrm{d}s_i, s_{-i}) \mathrm{d}s_{-i} \end{aligned}$$

In what follows, we proceed to show that a^* solves

$$\max_{a\in\Phi} \mathcal{L}(a), \tag{B.4}$$

which is sufficient for a^* to be a solution to (B.1).

Step 1: $\mathcal{L}$ is concave.

Note that for all s_{-i},

$$\int_0^1 \Big(\int_0^{s_i} a_i(\tilde{s}_i, s_{-i})\mathrm{d}\tilde{s}_i\Big)\Lambda_i(\mathrm{d}s_i, s_{-i}) = \int_0^1 a_i(s_i, s_{-i})\big(\Lambda_i(1, s_{-i}) - \Lambda_i(s_i, s_{-i})\big)\mathrm{d}s_i,$$
$$\int_0^1 -\frac{a_i(0, s_{-i})^2}{2}\Lambda_i(\mathrm{d}s_i, s_{-i}) = -\frac{a_i(0, s_{-i})^2}{2}(\Lambda_i(1, s_{-i}) - \Lambda_i(0, s_{-i})) = 0,$$

where the last equality comes from the construction of Λ_i. Hence, $\mathcal{L}(a)$ can be rewritten as

$$\begin{aligned}
\mathcal{L}(a) &= \iint \Big(u_0(a(s))f_1(s_1)f_2(s_2) - \sum_i a_i(s)(\Lambda_i(1, s_{-i}) - \Lambda_i(s_i, s_{-i}))\Big)\mathrm{d}s_1\mathrm{d}s_2 \\
&\quad + \sum_i \int_0^1\int_0^1 u_i(a_i(s), s_i)f_1(s_1)f_2(s_2)\mathrm{d}s_1\mathrm{d}s_2 \\
&\quad + \sum_i \int_0^1\int_0^1 \Big(s_i a_i(s) - \frac{a_i(s)^2}{2}\Big)\Lambda_i(\mathrm{d}s_i, s_{-i})\mathrm{d}s_{-i} \\
&= \iint \underbrace{\Big(u_0(a(s))f_1(s_1)f_2(s_2) - \sum_i a_i(s)(\Lambda_i(1, s_{-i}) - \Lambda_i(s_i, s_{-i}))\Big)}_{A(a,s)}\mathrm{d}s_1\mathrm{d}s_2 \qquad \text{(B.5)} \\
&\quad + \sum_i \int_0^1\int_0^1 \underbrace{\Big(u_i(a_i(s), s_i) - \kappa_i s_i a_i(s) + \kappa_i\frac{a_i(s)^2}{2}\Big)}_{B_i(a,s)} f_1(s_1)f_2(s_2)\mathrm{d}s_1\mathrm{d}s_2 \qquad \text{(B.6)} \\
&\quad + \sum_i \int_0^1\int_0^1 \underbrace{\Big(s_i a_i(s) - \frac{a_i(s)^2}{2}\Big)}_{C_i(a,s)}(\Lambda_i(\mathrm{d}s_i, s_{-i}) + \kappa_i f_{-i}(s_{-i})F_i(\mathrm{d}s_i))\mathrm{d}s_{-i}, \qquad \text{(B.7)}
\end{aligned}$$

where the second equality is obtained by simultaneously adding and subtracting the term $\sum_i \int_0^1\int_0^1 \big(\kappa_i s_i a_i(s_i, s_{-i}) - \kappa_i\frac{a_i(s_i,s_{-i})^2}{2}\big)f_1(s_1)f_2(s_2)\mathrm{d}s_1\mathrm{d}s_2$. For any s, $A(a, s)$ is concave in a because u_0 is concave. Hence, the integral in (B.5) is concave in a. For each i and s, $B_i(a, s)$ is also concave in a by the definition of κ_i. Hence, the term in (B.6) is concave in a. For any i and s, $C_i(a, s)$ is concave in a. Because we have already shown that $\Lambda_i(s_i, s_{-i})+\kappa_i f_{-i}(s_{-i}F_i(s_i))$ is increasing in s_i, $\Lambda_i(\mathrm{d}s_i, s_{-i})+\kappa_i f_{-i}(s_{-i}F_i(\mathrm{d}s_i))$ is in fact a positive measure. Hence, the term in (B.7) is also concave in a. Being the sum of functionals that are concave in a, $\mathcal{L}$ is also concave in a.

Step 2: For every $a \in \Phi$, $\lim_{\alpha\to 0} \frac{\mathcal{L}(\alpha a+(1-\alpha)a^*)-\mathcal{L}(a^*)}{\alpha} \le 0$.

For each $a \in \Phi$, using the expression of $\mathcal{L}(a)$ in the previous step, we can directly calculate the Gateaux derivative[1]

$$\begin{aligned}\partial\mathcal{L}(a) \equiv & \lim_{\alpha\to 0} \frac{\mathcal{L}(a^*+\alpha a)-\mathcal{L}(a^*)}{\alpha} \\ = & \sum_i \iint \Big(\frac{\partial w_i}{\partial a_i}(a_i^*(s), s) f_1(s_1) f_2(s_2) - (\Lambda_i(1, s_{-i}) - \Lambda_i(s))\Big) a_i(s) \mathrm{d}s_1 \mathrm{d}s_2 \\ & + \sum_i \iint \Big(s_i - a_i^*(s)\Big) a_i(s) \Lambda_i(\mathrm{d}s_i, s_{-i}) \mathrm{d}s_{-i}\end{aligned}$$

Recall that

$$\Lambda_i(1, s_{-i}) - \Lambda_i(s_i, s_{-i}) = \begin{cases} \kappa_i F_i(s_i) f_{-i}(s_{-i}), & \text{if } s_i \in [0, \underline{\phi}_i(s_{-i})], \\ \frac{\partial w_i}{\partial a_i}(s_i, s_i, s_{-i}) f_i(s_i) f_{-i}(s_{-i}), & \text{if } s_i \in (\underline{\phi}_i(s_{-i}), \bar{\phi}_i(s_{-i})), \\ -\kappa_i(1 - F_i(s_i)) f_{-i}(s_{-i}), & \text{if } s_i \in [\bar{\phi}_i(s_{-i}), 1], \end{cases}$$

and

$$a_i^*(s) = \begin{cases} \underline{\phi}_i(s_{-i}), & \text{if } s_i \in [0,\, \underline{\phi}_i(s_{-i})], \\ s_i, & \text{if } s_i \in (\underline{\phi}_i(s_{-i}),\, \bar{\phi}_i(s_{-i})), \\ \bar{\phi}_i(s_{-i}), & \text{if } s_i \in [\bar{\phi}_i(s_{-i}),\, 1]. \end{cases}$$

Hence, we can simplify the expression of $\partial\mathcal{L}(a)$ to

$$\begin{aligned}&\partial\mathcal{L}(a) \\ &= \sum_i \int_0^1 \Big[\underbrace{\int_0^{\underline{\phi}_i(s_{-i})} \Big(\frac{\partial w_i}{\partial a_i}(\underline{\phi}_i(s_{-i}), s) f_i(s_i) - \kappa_i F_i(s_i) - \kappa_i(s_i - \underline{\phi}_i(s_{-i})) f_i(s_i)\Big) a_i(s) \mathrm{d}s_i}_{\ell_i(a, s_{-i})}\Big] \mathrm{d}F_{-i} \\ &+ \sum_i \int_0^1 \Big[\underbrace{\int_{\bar{\phi}_i(s_{-i})}^1 \Big(\frac{\partial w_i}{\partial a_i}(\bar{\phi}_i(s_{-i}), s) f_i(s_i) + \kappa_i(1 - F_i(s_i)) - \kappa_i(s_i - \bar{\phi}_i(s_{-i})) f_i(s_i)\Big) a_i(s) \mathrm{d}s_i}_{h_i(a, s_{-i})}\Big] \mathrm{d}F_{-i}.\end{aligned}$$

[1] Let $f : [0,1]^2 \to \mathbb{R}$ be a continuously differentiable function, and μ be a finite measure over $[0,1]^2$. Then,

$$\begin{aligned}& \lim_{\alpha\to 0} \frac{\int_{[0,1]^2} f(a^*(s) + \alpha a(s)) \mu(\mathrm{d}s) - \int_{[0,1]^2} f(a^*(s)) \mu(\mathrm{d}s)}{\alpha} \\ =& \int_{[0,1]^2} \lim_{\alpha\to 0} \frac{f(a^*(s) + \alpha a(s)) - f(a^*(s))}{\alpha} \mu(\mathrm{d}s) \\ =& \int_{[0,1]^2} \Big(\sum_i \frac{\partial f}{\partial a_i}(a^*(s)) a_i(s)\Big) \mu(\mathrm{d}s),\end{aligned}$$

where the first equality comes from interchanging the order of limit and integration. This is guaranteed by the bounded convergence theorem.

Consider $\ell_i(a, s_{-i})$ first. Using the fact that $a_i(s)$ is increasing in s_i, we can also write $a_i(s) = a_i(\underline{\phi}_i(s_{-i}), s_{-i}) - \int_{[s_i, \underline{\phi}_i(s_{-i}))} a_i(\mathrm{d}s_i, s_{-i})$. Plugging this expression into $\ell_i(a, s_{-i})$, we obtain

$$
\begin{aligned}
&\ell_i(a, s_{-i}) \\
=& a_i(\underline{\phi}_i(s_{-i}), s_{-i}) \int_0^{\underline{\phi}_i(s_{-i})} \Big(\frac{\partial w_i}{\partial a_i}(\underline{\phi}_i(s_{-i}), s) f_i(s_i) - \kappa_i F_i(s_i) - \kappa_i (s_i - \underline{\phi}_i(s_{-i})) f_i(s_i) \Big) \mathrm{d}s_i \\
&- \int_{[0, \underline{\phi}_i(s_{-i}))} \Big[\int_0^{s_i} \Big(\frac{\partial w_i}{\partial a_i}(\underline{\phi}_i(s_{-i}), \tilde{s}) f_i(\tilde{s}_i) - \kappa_i F_i(\tilde{s}_i) - \kappa_i (\tilde{s}_i - \underline{\phi}_i(s_{-i})) f_i(\tilde{s}_i) \Big) \mathrm{d}\tilde{s}_i \Big] a_i(\mathrm{d}s_i, s_{-i}) \\
=& a_i(\underline{\phi}_i(s_{-i}), s_{-i}) \int_0^{\underline{\phi}_i(s_{-i})} \frac{\partial w_i}{\partial a_i}(\underline{\phi}_i(s_{-i}), s) f_i(s_i) \mathrm{d}s_i \\
&- \int_{[0, \underline{\phi}_i(s_{-i}))} \Big[\int_0^{s_i} \frac{\partial w_i}{\partial a_i}(\underline{\phi}_i(s_{-i}), \tilde{s}) f_i(\tilde{s}_i) \mathrm{d}\tilde{s}_i - \kappa_i (s_i - \underline{\phi}_i(s_{-i})) F_i(s_i) \Big] a_i(\mathrm{d}s_i, s_{-i}) \\
=& - \int_{[0, \underline{\phi}_i(s_{-i}))} \Big[\int_0^{s_i} \frac{\partial w_i}{\partial a_i}(\underline{\phi}_i(s_{-i}), \tilde{s}) f_i(\tilde{s}_i) \mathrm{d}\tilde{s}_i - \kappa_i (s_i - \underline{\phi}_i(s_{-i})) F_i(s_i) \Big] a_i(\mathrm{d}s_i, s_{-i}),
\end{aligned} \tag{B.8}
$$

where the first equality comes from changing the order of integration. The second equality comes from, for all s_i, $\int_0^{s_i} (\tilde{s}_i - \underline{\phi}_i(s_{-i})) f_i(\tilde{s}_i) \mathrm{d}\tilde{s}_i = (s_i - \underline{\phi}_i(s_{-i})) F_i(s_i) - \int_0^{s_i} F_i(\tilde{s}_i) \mathrm{d}\tilde{s}_i$. The third inequality comes from $\int_0^{\underline{\phi}_i(s_{-i})} \frac{\partial w_i}{\partial a_i}(\underline{\phi}_i(s_{-i}), s_i, s_{-i}) f_i(s_i) \mathrm{d}s_i = 0$ by condition C2. By condition C2 again, we know the term in the square bracket in (B.8) is nonnegative. This implies that $\ell_i(a, s_{-i}) \leq 0$. But notice that $a_i^*(s_i, s_{-i})$ is constant over $s_i \in [0, \underline{\phi}_i(s_{-i})]$. Therefore, $\ell_i(a^*, s_{-i}) = 0$.

Using a similar argument and condition C3, we can also show that $h_i(a, s_{-i}) \leq 0$ and $h_i(a^*, s_{-i}) = 0$. Therefore, we know $\partial \mathcal{L}(a) \leq 0$ for all $a \in \Phi$ and $\partial \mathcal{L}(a^*) = 0$.

Finally, using a similar argument as in the calculation of $\partial \mathcal{L}(a)$ (see footnote 1), we can calculate

$$
\lim_{\alpha \to 0} \frac{\mathcal{L}(\alpha a + (1-\alpha) a^*) - \mathcal{L}(a^*)}{\alpha} = \partial \mathcal{L}(a) - \partial \mathcal{L}(a^*) \leq 0.
$$

Step 3: a^* solves (B.4).

Suppose not. There exists $a \in \Phi$ such that $\mathcal{L}(a) > \mathcal{L}(a^*)$. By concavity from Step 1, $\mathcal{L}(\alpha a + (1-\alpha) a^*) \geq \alpha \mathcal{L}(a) + (1-\alpha) \mathcal{L}(a^*)$ for all $\alpha \in (0,1)$. Equivalently, $\frac{\mathcal{L}(\alpha a + (1-\alpha) a^*) - \mathcal{L}(a^*)}{\alpha} \geq \mathcal{L}(a) - \mathcal{L}(a^*)$ for all $\alpha \in (0,1)$. Letting α go to 0 yields $\lim_{\alpha \to 0} \frac{\mathcal{L}(\alpha a + (1-\alpha) a^*) - \mathcal{L}(a^*)}{\alpha} \geq \mathcal{L}(a) - \mathcal{L}(a^*) > 0$, contradicting Step 2. Therefore, a^* is a solution to (B.4), completing the proof. $\square$

Online Appendix C Proofs for Section 4

Proof of Proposition 2. We first verify that all the conditions needed in Theorem 2 are satisfied. For this, we only verify condition U1. All other conditions are straightforward.

We continue to use notation $\underline{g}_i(x, s_{-i})$ and $\bar{g}_i(x, s_{-i})$ defined in the proof of Lemma 3. Moreover, for notational simplicity, let $\tilde{\lambda}_i = \frac{\lambda_i}{\lambda_0}$ for $i = 1, 2$. Consider $\underline{g}_i(x, s_{-i})$. It is easy to calculate that

$$\begin{aligned}\frac{\partial \underline{g}_i(x, s_{-i})}{\partial x} &= -2\int_0^x \tilde{\lambda}_i F_i(s_i)\mathrm{d}s_i - 2F_i(x)(x - s_{-i}),\\ \frac{\partial^2 \underline{g}_i(x, s_{-i})}{\partial x^2} &= 2F_i(x)\left[\frac{f_i(x)}{F_i(x)}(s_{-i} - x) - (\tilde{\lambda}_i + 1)\right].\end{aligned}$$

When $s_{-i} = 0$, $\frac{\partial^2 \underline{g}_i(x,0)}{\partial x^2} < 0$ for $x \in (0, 1]$. Therefore, $\underline{g}_i$ is strictly concave and hence strictly quasi-concave. Assume $s_{-i} > 0$. Let $\theta(x) \equiv \frac{f_i(x)}{F_i(x)}(s_{-i} - x) - (\tilde{\lambda}_i + 1)$. Because $\frac{f_i}{F_i}$ is decreasing by Lemma 16, θ is strictly decreasing over $(0, s_{-i}]$. Because $\lim_{x\downarrow 0} \frac{f_i(x)}{F_i(x)} = +\infty$ by Lemma 16 again, we know $\lim_{x\downarrow 0} \theta(x) = +\infty$. Moreover, because $\theta(s_{-i}) < 0$, we know there exists $x' \in (0, s_{-i})$ such that θ is positive over $(0, x')$ and negative over (x', s_{-i}). Clearly, θ is also negative over $[s_{-i}, 1]$. Therefore, over the interval $(0, 1)$, $\frac{\partial^2 \underline{g}_i(\,\cdot\,, s_{-i})}{\partial x^2}$ single-crosses the x-axis from above, implying that $\underline{g}_i(\,\cdot\,, s_{-i})$ is strictly quasi-concave. We can similarly show that $\bar{g}_i(\,\cdot\,, s_{-i})$ is strictly quasi-concave.

From the proof of Lemma 3, we know that $c_i^*(s_{-i}) = \arg\max_{x\in[0,1]} \underline{g}_i(x, s_{-i})$. Observe that $\frac{\partial \underline{g}_i(0,s_{-i})}{\partial x} = 0$ for all s_{-i}. When $s_{-i} = 0$, the above analysis implies that $\frac{\partial \underline{g}_i(x,s_{-i})}{\partial x} < 0$ for $x > 0$. Therefore, $c_i^*(0) = 0$. When $s_i > 0$, the above analysis implies that $c_i^*(s_{-i}) > 0$ and satisfies the first order condition

$$\frac{\partial \underline{g}_i(c_i^*(s_{-i}), s_{-i})}{\partial x} = -2\int_0^{c_i^*(s_{-i})} \tilde{\lambda}_i F_i(s_i)\mathrm{d}s_i - 2F_i(c_i^*(s_{-i}))(c_i^*(s_{-i}) - s_{-i}) = 0,$$

or equivalently

$$c_i^*(s_{-i}) = s_{-i} - \tilde{\lambda}_i \frac{\int_0^{c_i^*(s_{-i})} F_i(s_i)\mathrm{d}s_i}{F_i(c_i^*(s_{-i}))} < s_{-i}. \tag{C.1}$$

Similarly, we can show that $d_i^*(1) = 1$. When $s_{-i} < 1$, we have $d_i^*(s_{-i}) < 1$ and is determined by

$$d_i^*(s_{-i}) = s_{-i} + \tilde{\lambda}_i \frac{\int_{d_i^*(s_{-i})}^1 (1 - F_i(s_i))\mathrm{d}s_i}{1 - F_i(d_i^*(s_{-i}))} > s_{-i}. \tag{C.2}$$

This completes the proof. □

Propositions 3 and 4 are built on the next two simple lemmas. Lemma C.1 is a technical result about log-concavity. It strengthens some of the results in Lemma 16.

Lemma C.1. *If f_i is log-concave, both $s_i \mapsto \int_0^{s_i} F_i(s_i')\mathrm{d}s_i'$ and $s_i \mapsto \int_{s_i}^1 (1 - F_i(s_i'))\mathrm{d}s_i'$ are strictly log-concave. Therefore, $\frac{F_i(s_i)}{\int_0^{s_i} F_i(s_i')\mathrm{d}s_i'}$ is strictly decreasing and $\frac{1-F_i(s_i)}{\int_{s_i}^1 (1-F_i(s_i'))\mathrm{d}s_i'}$ is strictly increasing.*

Proof. We only show that $s_i \mapsto \int_{s_i}^1 (1 - F(s_i'))\mathrm{d}s_i'$ is strictly log-concave. The other one is similar. Consider any $s_i \in (0, 1)$. By part (i) in Lemma 16, we know there exists $s_i'' \in (s_i, 1)$ such that

$$\frac{f_i(s_i)}{1 - F_i(s_i)} \le \frac{f_i(s_i')}{1 - F_i(s_i')}, \ \forall s_i' \in (s_i, 1),$$

with strictly inequality when $s_i' \in (s_i'', 1)$. This implies

$$\frac{f_i(s_i)}{1 - F_i(s_i)} \int_{s_i}^1 (1 - F_i(s_i'))\mathrm{d}s_i' < \int_{s_i}^1 \frac{f_i(s_i')}{1 - F_i(s_i')}(1 - F_i(s_i'))\mathrm{d}s_i' = 1 - F_i(s_i),$$

which in turn implies

$$\left[\log \int_{s_i}^1 (1 - F_i(s_i'))\mathrm{d}s_i'\right]'' = \frac{f_i(s_i)\int_{s_i}^1 (1 - F_i(s_i'))\mathrm{d}s_i' - (1 - F_i(s_i))^2}{\left(\int_{s_i}^1 (1 - F_i(s_i'))\mathrm{d}s_i'\right)^2} < 0.$$

Therefore, $\int_{s_i}^1 (1 - F_i(s_i'))\mathrm{d}s_i'$ is strictly log-concave. □

Lemma C.2 below shows the monotone comparative statics of agents' unilaterally constrained delegation rules with respect to the parameters. Denote by $(c^*_{i,\lambda_0,\lambda_i}, d^*_{i,\lambda_0,\lambda_i})$ the unilaterally constrained delegation rule for agent i when the importance of coordination is λ_0 and that of his adaptation is λ_i.[2]

Lemma C.2. *For any $s_{-i} \in (0, 1)$, $c^*_{i,\lambda_0,\lambda_i}(s_{-i})$ is strictly increasing in λ_0 and strictly decreasing in λ_i; $d^*_{i,\lambda_0,\lambda_i}(s_{-i})$ is strictly decreasing in λ_0 and strictly increasing in λ_i.*

Proof of Lemma C.2. For example, assume $\bar{\lambda}_i > \underline{\lambda}_i$. Pick any $s_{-i} \in (0, 1)$. For notational simplicity, let $\underline{c} = c^*_{i,\lambda_0,\underline{\lambda}_i}(s_{-i})$ and $\bar{c} = c^*_{i,\lambda_0,\bar{\lambda}_i}(s_{-i})$. By (C.1), we have

$$\underline{c} + \frac{\underline{\lambda}_i}{\lambda_0}\frac{\int_0^{\underline{c}} F_i(s_i)\mathrm{d}s_i}{F_i(\underline{c})} = \bar{c} + \frac{\bar{\lambda}_i}{\lambda_0}\frac{\int_0^{\bar{c}} F_i(s_i)\mathrm{d}s_i}{F_i(\bar{c})} > \bar{c} + \frac{\underline{\lambda}_i}{\lambda_0}\frac{\int_0^{\bar{c}} F_i(s_i)\mathrm{d}s_i}{F_i(\bar{c})}.$$

[2] The unilaterally constrained delegation rule for agent i does not depend on the importance of agent $-i$'s adaptation.

Because $c \mapsto c + \frac{\lambda_i}{\lambda_0}\frac{\int_0^c F_i(s_i)\mathrm{d}s_i}{F_i(c)}$ is strictly increasing by Lemma C.1, we know $\underline{c} > \bar{c}$. This proves that $c^*_{i,\lambda_0,\lambda_i}(s_{-i})$ is strictly decreasing in λ_i. The same argument can be applied to show that $c^*_{i,\lambda_0,\lambda_i}(s_{-i})$ is strictly increasing in λ_0. The proof for $d^*_{i,\lambda_0,\lambda_i}$ is analogous. □

Proof of Proposition 3. Let $(\phi^*_{1,\lambda_0}, \phi^*_{2,\lambda_0})$ be the principal's optimal contingent delegation when the importance of coordination to her is λ_0. For any s_{-i}, We show that $\underline{\phi}^*_{i,\lambda_0}(s_{-i})$ is increasing while $\bar{\phi}^*_{i,\lambda_0}(s_{-i})$ is decreasing in λ_0, for both $i = 1, 2$. For notational simplicity, we suppress λ_i from the previous notation $c^*_{i,\lambda_0,\lambda_i}$ and $d^*_{i,\lambda_0,\lambda_i}$, and directly write c^*_{i,λ_0} and d^*_{i,λ_0}.

Consider $0 < \underline{\lambda}_0 < \bar{\lambda}_0 < \infty$. We show $\bar{\phi}^*_{1,\bar{\lambda}_0} \leq \bar{\phi}^*_{1,\underline{\lambda}_0}$ and $\underline{\phi}^*_{2,\bar{\lambda}_0} \geq \underline{\phi}^*_{2,\underline{\lambda}_0}$. The proof is most easily understood by looking at Figure C.1. Let $(\bar{L}_{1,\lambda_0}, \underline{H}_{2,\lambda_0})$ be the intersection of d^*_{1,λ_0} and c^*_{2,λ_0} for $\lambda_0 \in \{\underline{\lambda}_0, \bar{\lambda}_0\}$. By Lemma C.2, we know $d^*_{1,\bar{\lambda}_0} \leq d^*_{1,\underline{\lambda}_0}$ and $c^*_{2,\bar{\lambda}_0} \geq c^*_{2,\underline{\lambda}_0}$. Hence in Figure C.1, $(\bar{L}_{1,\underline{\lambda}_0}, \underline{H}_{2,\underline{\lambda}_0})$ can only appear in one of the regions i, i, or iii .

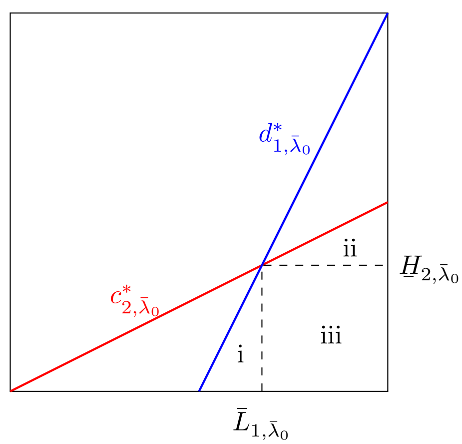


Figure C.1: Graph for the proof of Proposition 3

We claim that, in fact, $(\bar{L}_{1,\underline{\lambda}_0}, \underline{H}_{2,\underline{\lambda}_0})$ can only be in region iii. To see this, note that $c^*_{2,\lambda_0}(d^*_{1,\lambda_0}(\underline{H}_{2,\lambda_0})) = \underline{H}_{2,\lambda_0}$, for $\lambda_0 \in \{\underline{\lambda}_0, \bar{\lambda}_0\}$. Using (C.1), (C.2), and the fact $d^*_{1,\lambda_0}(\underline{H}_{2,\lambda_0}) = \underline{L}_{1,\lambda_0}$, we know

$$
\begin{aligned}
0 &= \frac{\lambda_2}{\underline{\lambda}_0}\frac{\int_{\bar{L}_{1,\underline{\lambda}_0}}^1 (1 - F_1(s_1))\mathrm{d}s_1}{1 - F_1(\bar{L}_{1,\underline{\lambda}_0})} - \frac{\lambda_1}{\underline{\lambda}_0}\frac{\int_0^{\underline{H}_{2,\underline{\lambda}_0}} F_2(s_2)\mathrm{d}s_2}{F_2(\underline{H}_{2,\underline{\lambda}_0})} \\
&= \frac{\lambda_2}{\bar{\lambda}_0}\frac{\int_{\bar{L}_{1,\bar{\lambda}_0}}^1 (1 - F_1(s_1))\mathrm{d}s_1}{1 - F_1(\bar{L}_{1,\bar{\lambda}_0})} - \frac{\lambda_1}{\bar{\lambda}_0}\frac{\int_0^{\underline{H}_{2,\bar{\lambda}_0}} F_2(s_2)\mathrm{d}s_2}{F_2(\underline{H}_{2,\bar{\lambda}_0})}.
\end{aligned}
$$

Because $x \mapsto \frac{\int_x^1 (1-F_1(s_1))\mathrm{d}s_1}{1-F_1(x)}$ is strictly decreasing and $x \mapsto \frac{\int_0^x F_2(s_2)\mathrm{d}s_2}{F_2(x)}$ is strictly increasing by Lemma C.1, it is easy to see from the above equation that we can have neither $\bar{L}_{1,\underline{\lambda}_0} \leq \bar{L}_{1,\bar{\lambda}_0}$ and $\underline{H}_{2,\underline{\lambda}_0} < \underline{H}_{2,\bar{\lambda}_0}$, nor $\bar{L}_{1,\underline{\lambda}_0} > \bar{L}_{1,\bar{\lambda}_0}$ and $\underline{H}_{2,\underline{\lambda}_0} \geq \bar{H}_{2,\underline{\lambda}_0}$. In other words, $(\bar{L}_{1,\underline{\lambda}_0},\, \underline{H}_{2,\underline{\lambda}_0})$ can be in neither region i nor region ii.

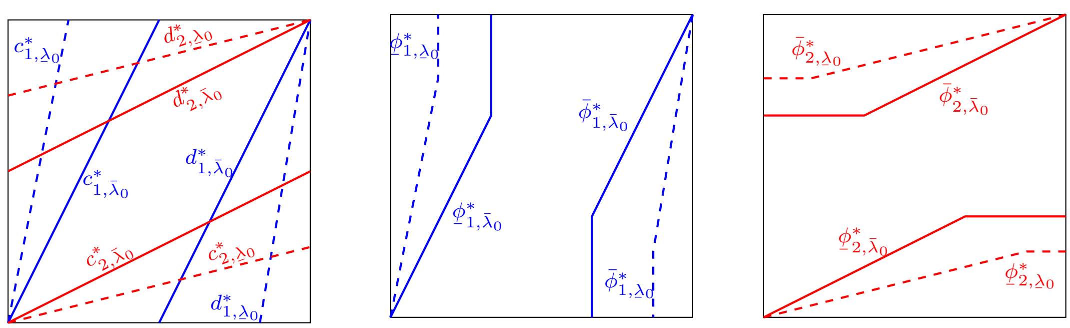
Therefore, $(\bar{L}_{1,\underline{\lambda}_0},\, \underline{H}_{2,\underline{\lambda}_0})$ is in region iii. Equivalently, $\bar{L}_{1,\underline{\lambda}_0} \geq \bar{L}_{1,\bar{\lambda}_0}$ and $\underline{H}_{2,\underline{\lambda}_0} \leq \underline{H}_{2,\bar{\lambda}_0}$. For any $s_2 \in [0,1)$, we then have

$$\bar{\phi}^*_{1,\underline{\lambda}_0}(s_1) = \max\{d^*_{1,\underline{\lambda}_0}(s_1),\, \bar{L}_{1,\underline{\lambda}_0}\} \geq \max\{d^*_{1,\bar{\lambda}_0}(s_1),\, \bar{L}_{1,\bar{\lambda}_0}\} = \bar{\phi}^*_{1,\bar{\lambda}_0}(s_1).$$

Similarly, for any $s_1 \in (0,1]$, we have

$$\underline{\phi}^*_{2,\underline{\lambda}_0}(s_2) = \min\{c^*_{2,\underline{\lambda}_0}(s_2),\, \underline{H}_{2,\underline{\lambda}_0}\} \leq \min\{c^*_{2,\bar{\lambda}_0}(s_2),\, \underline{H}_{2,\bar{\lambda}_0}\} = \underline{\phi}^*_{2,\bar{\lambda}_0}(s_2).$$

Figure C.2 gives an illustration. □



(a) $(c^*_i,\, d^*_i)$ (b) ϕ^*_1 (c) ϕ^*_2

Figure C.2: Importance of coordination and optimal discretion: $\bar{\lambda}_0 > \underline{\lambda}_0$

Proof of Proposition 4. It is a direct implication of Lemma C.2. See Figure C.3 for an illustration. □

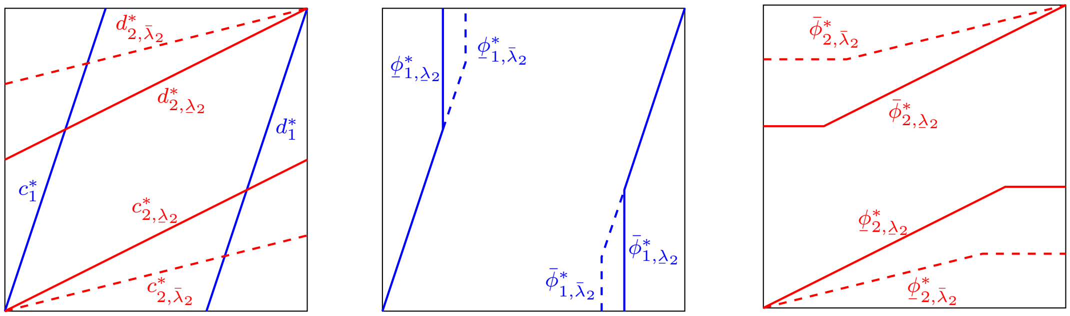
Proposition 5 is a direct implication of Lemma C.3 below. Denote by $(c^*_{i,f_i},\, d^*_{i,f_i})$ i's unilaterally coordinated delegation rule when his state distribution is f_i.

Lemma C.3. *Suppose $0 < \lambda_i < \infty$. Consider two densities $\underline{f}_i$ and $\bar{f}_i$ of agent i's state distribution. If the likelihood ratio $\bar{f}_i/\underline{f}_i$ is (strictly) increasing, then $c^*_{i,\bar{f}_i}(s_{-i}) \geq (>)\, c^*_{i,\underline{f}_i}(s_{-i})$ and $d^*_{i,\bar{f}_i}(s_{-i}) \geq (>)\, d^*_{i,\underline{f}_i}(s_{-i})$ for all $s_{-i} \in (0,1)$.*

Proof of Lemma C.3. Let $\bar{F}_i$ and $\underline{F}_i$ be the c.d.f.'s of $\bar{f}_i$ and $\underline{f}_i$ respectively. Because $\bar{f}_i$ and $\underline{f}_i$ satisfy the (strict) MLRP, we know that, for all $c, d \in (0,1)$,[3]

$$\frac{\int_0^c \bar{F}_i(s_i)\mathrm{d}s_i}{\bar{F}_i(c)} \leq (<) \frac{\int_0^c \underline{F}_i(s_i)\mathrm{d}s_i}{\underline{F}_i(c)} \text{ and } \frac{\int_d^1 (1-\bar{F}_i(s_i))\mathrm{d}s_i}{1-\bar{F}_i(s_i)} \geq (>) \frac{\int_d^1 (1-\underline{F}_i(s_i))\mathrm{d}s_i}{1-\underline{F}_i(s_i)}.$$

[3]See, for example, Theorem 1.C.1 in Shaked and Shanthikumar (2007).



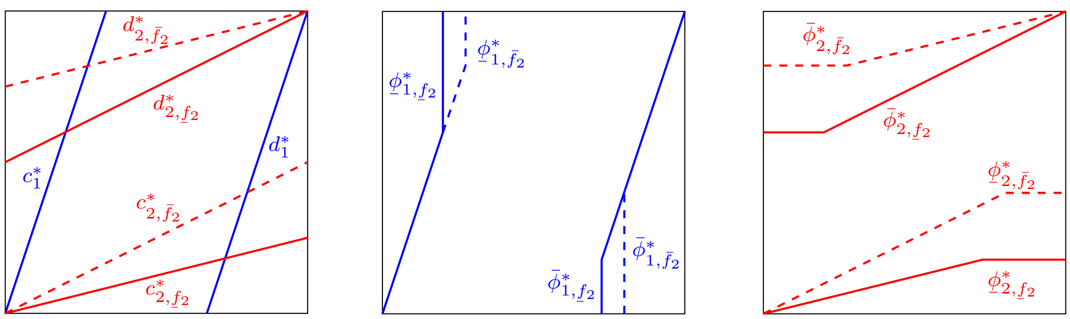
(a) $(c_i^*,\, d_i^*)$ (b) ϕ_1^* (c) ϕ_2^*

Figure C.3: Relative importance and optimal discretion: $\bar{\lambda}_2 > \underline{\lambda}_2$

Consider $s_{-i} \in (0,1)$. Let $\underline{c} = c^*_{i,\underline{f}_i}(s_{-i})$ and $\bar{c} = c^*_{i,\bar{f}_i}(s_{-i})$. By (C.1), we have

$$\underline{c} + \frac{\lambda_i}{\lambda_0}\frac{\int_0^{\underline{c}} \underline{F}_i(s_i)\mathrm{d}s_i}{\underline{F}_i(\underline{c})} = \bar{c} + \frac{\lambda_i}{\lambda_0}\frac{\int_0^{\bar{c}} \bar{F}_i(s_i)\mathrm{d}s_i}{\bar{F}_i(\bar{c})} \leq (<)\, \bar{c} + \frac{\lambda_i}{\lambda_0}\frac{\int_0^{\bar{c}} \underline{F}_i(s_i)\mathrm{d}s_i}{\underline{F}_i(\bar{c})}.$$

Again, because $c \mapsto c + \frac{\lambda_i}{\lambda_0}\frac{\int_0^{c} \underline{F}_i(s_i)\mathrm{d}s_i}{\underline{F}_i(c)}$ is strictly increasing, we know $\underline{c} \leq (<)\, \bar{c}$. Figure C.4 provides an illustration. □



(a) $(c_i^*,\, d_i^*)$ (b) ϕ_1^* (c) ϕ_2^*

Figure C.4: State distribution and optimal discretion: $\bar{f}_2/\underline{f}_2$ is increasing

References

Shaked, M. and J. G. Shanthikumar (2007): *Stochastic Orders*, Springer Science & Business Media.