Online Appendix

"Consumer Search and Optimal Information"

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This online appendix contains the missing proofs of the article. Appendix A provides the proof of Corollary 1. Appendices B to E contain the proofs of Propositions 2 to 4. Appendix F shows that it is without loss of generality to focus on consumers' pure strategy. Appendix G proves Proposition 6.

Appendix A Proof of Corollary 1

To prove Corollary 1, we need the following two claims, which are also needed in later proofs.

Claim A.1. *If $x > b$, $h(x; a, b, v)$ is strictly increasing in v.*

Proof. The easiest way to see this is to note $h_{xv}(x; a, b, v) = \frac{\rho(x-b)}{(x-v)^2} > 0$. If $v' > v$, we have $h(x; a, b, v') = s + \int_b^x h_x(\tilde{x}; a, b, v')\mathrm{d}\tilde{x} > s + \int_b^x h_x(\tilde{x}; a, b, v)\mathrm{d}\tilde{x} = h(x; a, b, v)$. $\square$

Claim A.2. *If $c_{a,b,v} \in \mathcal{U}$, then $c_{a,b',a} \in \mathcal{U}$ for all $b' \in [\mu - s,\, b]$. Moreover, $c_{0,\mu-s,0} \in \mathcal{U}$.*

Proof. Note that $c_{a,b,v}$ and $c_{a,b,a}$ differ from each other only over $(b, 1]$. Because $v \geq a$, $c_{a,b,a} \leq c_{a,b,v}$ over $(b, 1]$ by Claim A.1, implying $c_{a,b,a} \in \mathcal{U}$. Consider $b' \in [\mu - s,\, b)$. For $x \in [a, b']$, $c_{a,b',a}(x) = \mu - a - \frac{\mu-s-a}{b'-a}(x - a) < \mu - a - \frac{\mu-s-a}{b-a}(x - a) = c_{a,b,a}(x)$. For $x \in (b', b)$, $c_{a,b',a}(x) < s < c_{a,b,a}(x)$. For $x \in [b, 1]$, $h(x; a, b', a) = s - (\mu - s - a)\log\frac{x-a}{b'-a} < s - (\mu - s - a)\log\frac{x-a}{b-a} = h(x; a, b, a)$, implying $c_{a,b',a}(x) \leq c_{a,b,a}(x)$. Therefore, $c_{a,b',a} \leq c_{a,b,a}$, implying $c_{a,b',a} \in \mathcal{U}$. Finally, because $c_{a,\mu-s,a} = c_{0,\mu-s,a}$, we know $c_{0,\mu-s,0} \in \mathcal{U}$ by Claim A.1 again. $\square$

Proof of Corollary 1. By Claim A.2, the search market (F, s) admits active search if and only if $c_{0,\mu-s,0}$ is feasible, which is equivalent to

$$s - (\mu - s)\log\frac{x}{\mu - s} \leq c_F(x),\ \forall x \in [\mu - s, 1]. \tag{A.1}$$

Because $F \neq F_0$, F places positive mass over $[0, \mu)$. Thus, we know $c_F(\mu) > 0$. This implies $-\mu \log\frac{x}{\mu} < c_F(x)$ over $[\mu, 1]$. By uniform continuity, we know (A.1) must be satisfied for positive but small s. Thus, there does exist search cost $s > 0$ such that (F, s) admits active search. Let $s^* > 0$ be the least upper bound of such s's. Because $\lim_{s\uparrow\mu} s - (\mu - s)\log\frac{\mu}{\mu-s} = \mu > c_F(\mu)$, we know $s^* < \mu$. For any $s < s^*$, there exists $s < s' < s^*$ such that (F, s') admits active search. Because $s - (\mu - s)\log\frac{x}{\mu-s} < s' - (\mu - s')\log\frac{x}{\mu-s'} \leq c_F(x)$ for all $x \in [\mu - s,\ 1]$, we know (F, s) admits active search too. Finally, for any $x > \mu - s^*$, we have $x \geq \mu - s$ when s is sufficiently close to s^*. Hence, $s^* - (\mu - s^*)\log\frac{x}{\mu-s^*} = \lim_{s\uparrow s^*} s - (\mu - s)\log\frac{x}{\mu-s} \leq c_F(x)$. This implies (F, s^*) admits active search too. Therefore, (F, s) admits active search if and only if $s \in (0, s^*]$, completing the proof. $\square$

Before we move to the next section, we establish a strict single-crossing property of two h curves under different parameters. This is particularly useful in reducing the infinitely many feasibility constraints into a single one. We need it in the proofs of Propositions 3 and 4.

Claim A.3. *Consider two sets of parameters (a, b, v) and (a', b', v'). Suppose $b' < b$ and $v' \geq v$. Over $[b, 1]$, $h(x; a', b', v')$ crosses $h(x; a, b, v)$ at most once and from below. That is, one and only one of the following holds:*

(i) $h(x; a', b', v') < h(x; a, b, v)$ for all $x \in [b, 1]$.

(ii) there exists $\hat{x} \in (b, 1]$ such that $h(x; a', b', v') < (>)\, h(x; a, b, v)$ if $x < \hat{x}$ $(x > \hat{x})$.

Consequently, for any $x' \in [b, 1]$, $h(x; a', b', v') \leq h(x; a, b, v)$ for all $x \in [b, x']$ if and only if $h(x'; a', b', v') \leq h(x'; a, b, v)$.

Proof. If $h(x; a', b', v')$ and $h(x; a, b, v)$ do not intersect over the interval $[b, 1]$, then (i) holds, as $h(b; a, b', v') < s = h(b; a, b, v)$.

Suppose they intersect at some $\hat{x} \in (b, 1]$. For clarity, let $\pi = \frac{\mu - s - a}{b - a}(b - v)$ and $\pi' = \frac{\mu - s - a'}{b' - a'}(b' - v')$. Because $h(b; a', b', v') < h(b; a, b, v)$ and $h(\hat{x}; a', b', v') = h(\hat{x}; a, b, v)$, there exists $\tilde{x} \in (b, \hat{x})$ such that $h_x(\tilde{x}; a', b', v') > h_x(\tilde{x}; a, b, v)$, or equivalently, $-\frac{\pi'}{\tilde{x} - v'} > -\frac{\pi}{\tilde{x} - v}$. Because $v' \geq v$, it is straightforward to verify $-\frac{\pi'}{x - v'} > -\frac{\pi}{x - v}$ for all $x > \tilde{x}$, or equivalently $h_x(x; a', b', v') > h_x(x; a, b, v)$ for all $x > \tilde{x}$. This immediately implies $h(x; a', b', v') > h(x; a, b, v)$ for all $x > \hat{x}$. If there exists $x \in (b, \hat{x})$ such that $h(x; a', b', v') > h(x; a, b, v)$, then there exists $\hat{\hat{x}} \in (b, x)$ at which the two curves intersect. Applying the same argument, we can obtain $h(\hat{x}; a', b', v') > h(\hat{x}; a, b, v)$, a contradiction. Therefore, $h(x; a', b', v') < h(x; a, b, v)$ for $x < \hat{x}$. $\square$

Appendix B Proof of Proposition 2

B.1 Existence

Existence is a direct implication of the following claim.

Claim B.1. *Let*

$$K \equiv \{(a, b, v) \in \mathbb{R}^3 \,|\, a \in [0,\, \mu - s), b \in [\mu - s,\, \bar{b}], v \in [a, b), \text{ and } c_{a,b,v} \leq c_F\} \quad \text{(B.1)}$$

be the set of feasible parameters (a, b, v). Then, K is compact.

Proof. As K is clearly bounded, we only need to show it is closed. Assume $(a_n, b_n, v_n)_{n \geq 1}$ is a convergent sequence in K and let (a, b, v) be its limit. We proceed to show $(a, b, v) \in K$.

First, we show $v \in [a, b)$. Note that, for all $n \geq 1$, $h(1; a_n, b_n, v_n) = s - \frac{\mu - s - a_n}{b_n - a_n}(b_n - v_n) \log \frac{1 - v_n}{b_n - v_n} \geq s - (b_n - v_n) \log \frac{1 - v_n}{b_n - v_n}$. If $v = b$, we have $h(1; a_n, b_n, v_n) > 0$ for sufficiently large n. This contradicts $c_{a_n, b_n, v_n}(1) = \max\{h(1; a_n, b_n, v_n), 0\} \leq c_F(1) = 0$. Therefore, we must have $v < b$.

Second, we show $a \in [0, \mu - s)$. Suppose $a = \mu - s$ by contradiction. If $b > \mu - s$, we have $\lim_{n \to \infty} h(1; a_n, b_n, v_n) = \lim_{n \to \infty} s - \frac{\mu - s - a_n}{b_n - a_n}(b_n - v_n) \log \frac{1 - v_n}{b_n - v_n} = s > 0$. This, again, contradicts $c_{a_n, b_n, v_n}(1) \leq c_F(1)$. Thus, we must have $b = \mu - s$. But because $a_n \leq v_n < b_n$ for all n, we have $v = \lim_n v_n = \mu - s = b$, which contradicts the previous step. Therefore, we must have $a < \mu - s$.

Finally, we show $c_{a,b,v} \leq c_F$. The previous two steps guarantee that $c_{a,b,v}$ is well-defined. By construction, this is true for $x \in [0, a] \cup \{b\}$. If $x \in (a, b)$, we know $x \in (a_n, b_n)$ for sufficiently large n. Thus, $c_{a,b,v}(x) = s - \frac{\mu - s - a}{b - a}(x - b) = \lim_{n \to \infty} s - \frac{\mu - s - a_n}{b_n - a_n}(x - b_n) = \lim_{n \to \infty} c_{a_n, b_n, v_n}(x) \leq c_F(x)$. Similarly, if $x \in (b, 1]$, we know $x \in (b_n, 1]$ for sufficiently large n. Thus, $c_{a,b,v}(x) = \max\{h(x; a, b, v), 0\} = \lim_{n \to \infty} \max\{h(x; a_n, b_n, v_n), 0\} = \lim_{n \to \infty} c_{a_n, b_n, v_n}(x) \leq c_F(x)$. This completes the proof. $\square$

B.2 Uniqueness

Claim A.2 has shown that the set of achievable total welfare levels takes the form of either $[\mu - s, \hat{b})$ or $[\mu - s, \hat{b}]$ for some $\mu - s \leq \hat{b} \leq \bar{b}$. Claim B.1 implies that it must be $[\mu - s, \hat{b}]$. For every $b \in [\mu - s, \hat{b}]$, define

$$v(b) \equiv \max_{a \in [0, \mu - s]} \max_{v \in [a, b)} v \quad \text{subject to } c_{a,b,v} \leq c_F. \tag{B.2}$$

By Claim B.1 again, we know $v(b)$ is well defined. Note that the optimal consumer surplus is simply $\max_{b \in [\mu - s, \hat{b}]} v(b)$. We show the desired uniqueness by showing that (i) (B.2) has a unique solution for every $b \in [\mu - s, \hat{b}]$ (Claim B.2), and (ii) $v(b)$ is strictly concave (Claim B.3).

For every $b \in [\mu - s,\, \hat{b}]$, let $a(b)$ be the smallest $a \in [0,\, \mu - s)$ such that

$$s - \frac{\mu - s - a}{b - a}(x - b) \leq c_F(x),\ \forall x \in [a, b]. \tag{B.3}$$

Figure B.1 provides an illustration of the determination of $a(b)$. When $a = a(b)$, some of the constraints in (B.3) must be binding. In fact, $a(b)$ is the *minimal feasible low match value atom for* b. This is because any $c_{a,b,v}$ with $a < a(b)$ must be infeasible.

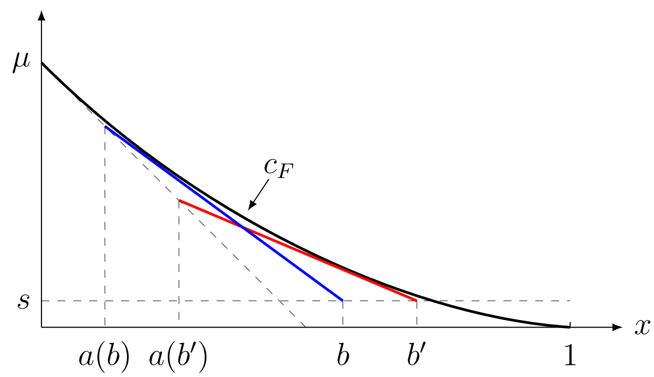


Figure B.1: Minimal feasible low match value atom

Claim B.2. *If $c_{a,b,v} \in \mathcal{U}$ and $a > a(b)$, then there exists $v' > v$ such that $c_{a(b),b,v'} \in \mathcal{U}$. Therefore, (B.2) has a unique solution and the optimal choice of a is $a(b)$.*

Proof. Because $\frac{\mu-s-a(b)}{b-a(b)} > \frac{\mu-s-a}{b-a}$, it is straightforward to verify $h(x; a(b), b, v) \leq h(x; a, b, v)$ for $x \in [b, 1]$, with equality if and only if $x = b$. Note $h_x(b; a(b), b, v) = -\frac{\mu-s-a(b)}{b-a(b)} < -\frac{\mu-s-a}{b-a} = h_x(b; a, b, v)$. By uniform continuity of h_x, there exists $\hat{x} > b$ and $v_1 > v$ such that $h_x(x; a(b), b, v') \leq h_x(x; a, b, v)$ for all $x \in [b, \hat{x}]$ and $v' \in [v, v_1]$. Therefore, for any $v' \in [v, v_1]$, $h(x; a(b), b, v') = s + \int_b^x h_x(\tilde{x}, a(b), b, v')\mathrm{d}\tilde{x} \leq s + \int_b^x h_x(\tilde{x}, a, b, v)\mathrm{d}\tilde{x} = h(x; a, b, v)$ for all $x \in [b, \hat{x}]$.

On the other hand, because $h(x; a(b), b, v) < h(x; a, b, v)$ for all $x \in [\hat{x}, 1]$, uniform continuity of h implies that there exists $v_2 > v$ such that for all $v' \in [v, v_2]$, $h(x; a(b), b, v') \leq h(x; a, b, v)$ for all $x \in [\hat{x}, 1]$. Pick any $v' \in (v, \min\{v_1, v_2\})$, we then know $h(x; a(b), b, v') \leq h(x; a, b, v) \leq c_F(x)$ for all $x \in [b, 1]$. By construction of $a(b)$, $c_{a(b),b,v'} \leq c_F(x)$ for all $x \in [0, b]$. Therefore, $c_{a(b),b,v'}$ is feasible. □

For $b \in [\mu - s, \hat{b}]$, let $\rho(b) \equiv \frac{\mu - s - a(b)}{b - a(b)}$ be the *maximal feasible probability of trade for* b. From Figure B.1, it is easy to note that $\rho(b)$ is strictly decreasing, as the blue line is steeper than the red line.[1]

Claim B.3. *$v(b)$ is strictly concave.*

Proof. Pick $b_1, b_2 \in [\mu - s, \hat{b}]$ and assume $b_1 < b_2$. Let $a_1 = a(b_1)$ and $a_2 = a(b_2)$. As $a(b)$ is increasing, we know $a_1 \leq a_2$. Let $v_1 = v(b_1)$ and $v_2 = v(b_2)$. By Claim B.2, both c_{a_1,b_1,v_1} and c_{a_2,b_2,v_2} are feasible. These two curves have a unique intersection, denoted by (x^*, y^*), over the interval $[a_2, b_1]$. Figure B.2 illustrates this intersection for the case $b_1 > \mu - s$. Panels (a) and (b) depict two different sub-cases, $a_1 < a_2$ and $a_1 = a_2$, respectively. In both panels, the two blue lines represent c_{a_1,b_1,v_1} and c_{a_2,b_2,v_2} over the relevant ranges. If $b_1 = \mu - s$, then $a_1 = 0$ and it is possible that $a_2 > 0$. However, because $c_{0,\mu-s,v_1} = c_{a_2,\mu-s,v_1}$, it is treated as the case $a_1 = a_2$. Note that the probability of trade $\rho(b_i) = \frac{\mu - s - a_i}{b_i - a_i}$ under c_{a_i,b_i,v_i} can be expressed in terms of (x^*, y^*): $\rho(b_i) = \frac{y^* - s}{b_i - x^*}$, for $i = 1, 2$.

Pick any $\lambda \in (0, 1)$ and let $b^\lambda = \lambda b_1 + (1 - \lambda) b_2$. Let a^λ be the unique solution to $s - \frac{y^* - s}{b^\lambda - x^*}(x - b^\lambda) = \mu - x$. In fact, a^λ is just the intersection of two straight lines: (i) the line that passes through (x^*, y^*) and (b^λ, s), and (ii) the downward-sloping 45-degree line. The red lines in both panels of Figure B.2 illustrate the former. For example, if $a_1 = a_2$, then a^λ just coincides with them. See panel (b). If $a_1 < a_2$, then $a^\lambda \in (a_1, a_2)$ and in general is not equal to $\lambda a_1 + (1 - \lambda) a_2$. See panel (a).

Let $v^\lambda = \lambda v_1 + (1 - \lambda) v_2$. We proceed to show $v(b^\lambda) > v^\lambda$. For this, it suffices to show that there exists $v' > v^\lambda$ such that $c_{a^\lambda,b^\lambda,v'}$ is feasible. This involves several

[1]By Claim B.2, the minimal feasible price given total welfare b can be written as $\min\{p \,|\, h(x; a(b), b, b - p) \leq c_F(x) \, \forall x \in [b, 1]\}$. Note $h(x; a(b), b, b - p) = s - \rho(b) p \log \frac{x - (b - p)}{p}$. Because $\rho(b)$ is decreasing, it is easy to see that $h(x; a(b), b, b - p)$ is increasing in b. By Claim A.1, it is then easy to see that the minimal feasible price is increasing in b.

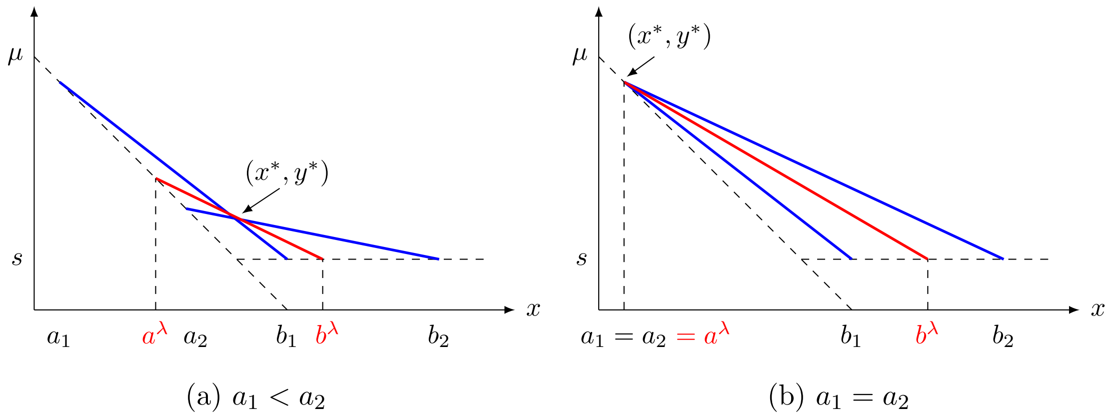


Figure B.2: Illustration of $c_{a^\lambda,b^\lambda,v^\lambda}$ over interval $[a^\lambda,\, b^\lambda]$

steps.

Step 1: $h(x; a^\lambda, b^\lambda, v^\lambda) \le c_F(x)$, for $x \in (b_2, 1]$.

For $x \in (b_2, 1]$, we have

$$\begin{aligned}
&h(x; a^\lambda, b^\lambda, v^\lambda)\\
=&s - \frac{y^* - s}{b^\lambda - x^*}(b^\lambda - v^\lambda)\log\frac{x - v^\lambda}{b^\lambda - v^\lambda}\\
=&s - \frac{y^* - s}{b^\lambda - x^*}(b^\lambda - v^\lambda)\log\left[\frac{\lambda(b_1 - v_1)}{b^\lambda - v^\lambda}\frac{x - v_1}{b_1 - v_1} + \frac{(1-\lambda)(b_2 - v_2)}{b^\lambda - v^\lambda}\frac{x - v_2}{b_2 - v_2}\right]\\
\le&s - \frac{\lambda(y^* - s)}{b^\lambda - x^*}(b_1 - v_1)\log\frac{x - v_1}{b_1 - v_1} - \frac{(1-\lambda)(y^* - s)}{b^\lambda - x^*}(b_2 - v_2)\log\frac{x - v_2}{b_2 - v_2}\\
=&\frac{\lambda(b_1 - x^*)}{b^\lambda - x^*}h(x; a_1, b_1, v_1) + \frac{(1-\lambda)(b_2 - x^*)}{b^\lambda - x^*}h(x; a_2, b_2, v_2)\\
\le&\frac{\lambda(b_1 - x^*)}{b^\lambda - x^*}c_{a_1,b_1,v_1}(x) + \frac{(1-\lambda)(b_2 - x^*)}{b^\lambda - x^*}c_{a_2,b_2,v_2}(x) \le c_F(x),
\end{aligned}\tag{B.4}$$

where the first inequality comes from concavity of the logarithm function.

Step 2: $h(x; a^\lambda, b^\lambda, v^\lambda) < c_F(x)$ for $x \in (b_2, 1]$.

Suppose not. There exists $\hat{x} \in (b_2, 1]$ such that $h(\hat{x}; a^\lambda, b^\lambda, v^\lambda) = c_F(\hat{x})$. Then, at this $\hat{x}$ all the inequalities in (B.4) must be equalities. For the first inequality to be an equality, we must have

$$\frac{\hat{x} - v_1}{b_1 - v_1} = \frac{\hat{x} - v_2}{b_2 - v_2}, \tag{B.5}$$

because the logarithm function is strictly concave. For the other inequalities to be equalities, we must have

$$h(\hat{x}; a_1, b_1, v_1) = h(\hat{x}; a_2, b_2, v_2). \tag{B.6}$$

Because $h(x; a_i, b_i, v_i) = s - \rho(b_i)(b_i - v_i) \log \frac{x - v_i}{b_i - v_i}$ for $i = 1, 2$, equations (B.5) and (B.6) together imply that the equilibrium expected profits of a matched firm under c_{a_1,b_1,v_1} and c_{a_2,b_2,v_2} are equal: $\rho(b_1)(b_1 - v_1) = \rho(b_2)(b_2 - v_2) = \pi$. This observation has two implications. First, because $b_1 < b_2$ by assumption, we know $\rho(b_1) > \rho(b_2)$. Thus, we must have $b_1 - v_1 < b_2 - v_2$. Second, because $h_x(x; a_i, b_i, v_i) = -\frac{\pi}{x - v_i}$ for $i = 1, 2$, the slopes of these two curves, $h(x; a_1, b_1, v_1)$ and $h(x; a_2, b_2, v_2)$, are uniformly ranked over $(b_2, 1]$. Because $h(b_2; a_1, b_1, v_1) < s = h(b_2; a_2, b_2, v_2)$, for (B.6) to hold, we must have $-\frac{\pi}{x - v_1} > -\frac{\pi}{x - v_2}$ for $x > b_2$, implying $v_1 < v_2$. Combining $b_1 - v_1 < b_2 - v_2$ and $v_1 < v_2$ implies $\frac{\hat{x} - v_1}{b_1 - v_1} > \frac{\hat{x} - v_1}{b_2 - v_2} > \frac{\hat{x} - v_2}{b_2 - v_2}$, which contradicts (B.5). Therefore, $h(x; a^\lambda, b^\lambda, v^\lambda) < c_F(x)$ for all $x \in (b_2, 1]$.

Step 3: there exists $v' > v^\lambda$ such that $h(x; a^\lambda, b^\lambda, v') \leq c_F(x)$ for $x \in [b^\lambda, 1]$.

For $x \in [b^\lambda, b_2)$, $h(x; a^\lambda, b^\lambda, v^\lambda) \leq s < c_{a_2,b_2,v_2}(x) \leq c_F(x)$. See Figure B.2 for the strict inequality. For $x = b_2$, $h(b_2; a^\lambda, b^\lambda, v^\lambda) < s = c_{a_2,b_2,v_2}(b_2) \leq c_F(b_2)$. These observations, together with Step 2, imply $h(x; a^\lambda, b^\lambda, v^\lambda) < c_F(x)$ for all $x \in [b^\lambda, 1]$. Then, by uniform continuity, there exists $v' > v^\lambda$ such that $h(x; a^\lambda, b^\lambda, v') \leq c_F(x)$ for all $x \in [b^\lambda, 1]$.

Step 4: $c_{a^\lambda,b^\lambda,v'}$ is feasible and therefore $v(b^\lambda) \geq v' > v^\lambda$.

By Step 3, $c_{a^\lambda,b^\lambda,v'} \leq c_F$ over $[b^\lambda, 1]$. For $x \in [a^\lambda, b^\lambda]$, Figure B.2 shows $c_{a^\lambda,b^\lambda,v'}(x) \leq \max\{c_{a_1,b_1,v_1}(x), c_{a_2,b_2,v_2}(x)\} \leq c_F(x)$. Hence, $c_{a^\lambda,b^\lambda,v'}$ is feasible, completing the proof.

□

Appendix C Proof of Proposition 3

In this and the next sections, the primary concern is how a change in search cost affects the determination of the consumer-optimal signal distribution. Because search cost s is also a parameter of h function, we explicitly add it as an argument and write $h(x; a, b, v, s)$. We do the same for $v(b)$, $a(b)$, and $\rho(b)$ defined in Appendix B.2, and write them as $v(b, s)$, $a(b, s)$, and $\rho(b, s)$, respectively. We also write $c^s_{a,b,v}$ to denote an incremental benefit function designed for search cost s.[2] Throughout this and the next sections, we fix $0 < \underline{s} < \bar{s} < s^*$.

C.1 Proof of part (i)

Part (i) is a direct corollary of the following claim, which will also be used in the proof of part (ii).

Claim C.1. *Suppose $c^{\bar{s}}_{a,b,v}$ is feasible under search cost $\bar{s}$. Let $b' = b + \frac{\bar{s}-\underline{s}}{\rho}$, where $\rho = \frac{\mu-\bar{s}-a}{b-a}$ is the probability of trade under $c^{\bar{s}}_{a,b,v}$. Then, there exists $v' > v$ such that $c^{\underline{s}}_{a,b',v'}$ is feasible under search cost $\underline{s}$.*

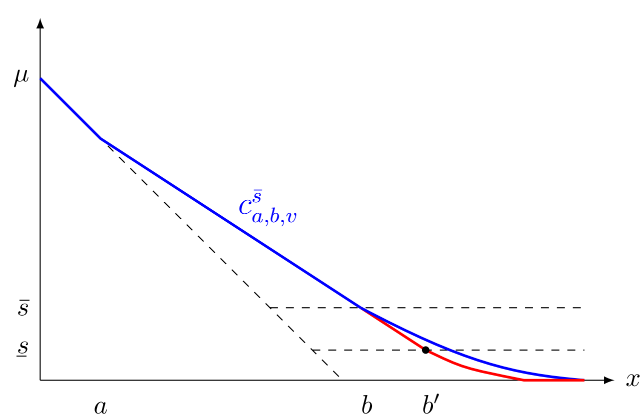


Figure C.1: Proof of Claim C.1

[2] Note that $c^s_{a,b,v}$ and $c^{s'}_{a,b,v}$ are different if $s \neq s'$, even though the parameters (a, b, v) are the same.

Proof. The proof is most easily understood by looking at Figure C.1. The blue curve represents $c^{\bar{s}}_{a,b,v}$. By construction, it is a straight line over $[a, b]$ with slope $-\rho$. The red line over $[b, b']$ is its extension, which intersects the horizontal line of value $\underline{s}$ at exactly $b' = b + \frac{\bar{s}-\underline{s}}{\rho}$.

Consider the curve $h(x; a, b', v, \underline{s})$ and compare it to $h(x; a, b, v, \bar{s})$. At $x = b'$, $h(b'; a, b', v, \underline{s}) = \underline{s} < c^{\bar{s}}_{a,b,v}(b') = h(b'; a, b, v, \bar{s})$, as is depicted in Figure (C.1). For $x > b'$, we have $h_x(x; a, b', v, \underline{s}) = -\frac{\rho(b'-v)}{x-v} = -\frac{\rho(b-v)}{x-v} - \frac{\bar{s}-\underline{s}}{x-v} < -\frac{\rho(b-v)}{x-v} = h_x(x; a, b, v, \bar{s})$. Therefore, $h(x; a, b', v, \underline{s}) < h(x; a, b, v, \bar{s})$ for all $x \in [b', 1]$. By uniform continuity, there exists $v' > v$ such that $h(x; a, b', v', \underline{s}) \leq h(x; a, b, v, \bar{s})$ for all $x \in [b', 1]$. This immediately implies that $c^{\underline{s}}_{a,b',v'}$ is feasible under search cost $\underline{s}$. $\square$

C.2 Proof of part (ii)

Claim C.2. *If $v(b, \bar{s})$ is increasing over an interval $[b_1, b_2]$, then $v(b, \underline{s})$ is increasing over the interval $[b_1 + \frac{\bar{s}-\underline{s}}{\rho(b_1,\bar{s})},\ b_2 + \frac{\bar{s}-\underline{s}}{\rho(b_2,\bar{s})}]$.*

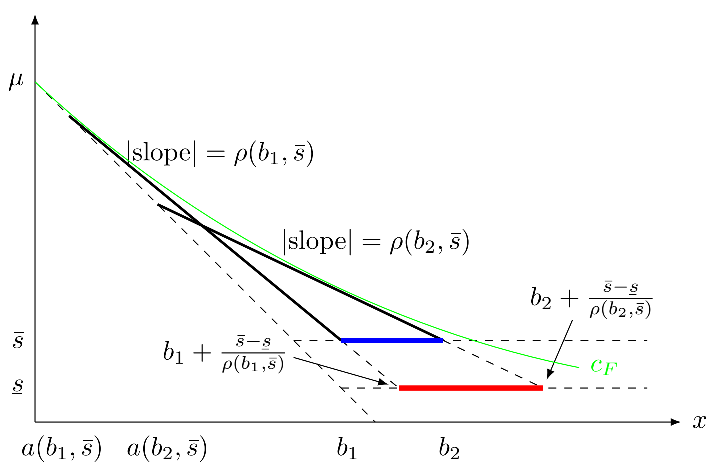


Figure C.2: Illustration of the intervals in Claim C.2

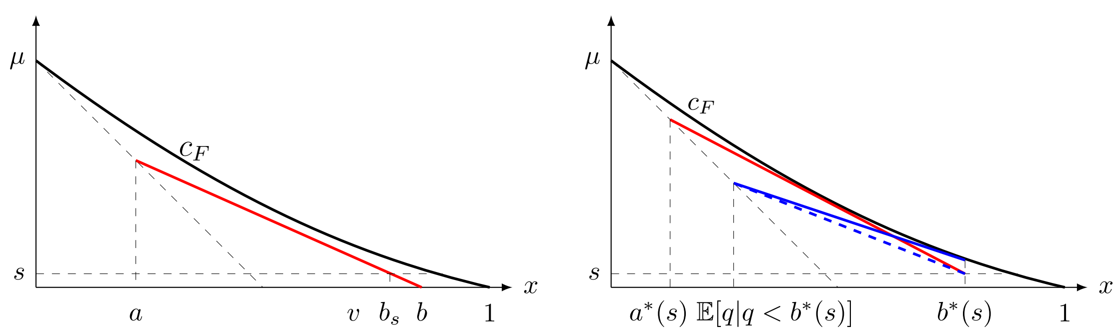
Figure C.2 is helpful in understanding Claim C.2. Focus on the larger search cost $\bar{s}$ first and consider an interval $[b_1, b_2]$. It is described by the blue interval. Recall that $a(b, \bar{s})$ is the minimal feasible low match value atom and $\rho(b, \bar{s}) = \frac{\mu-\bar{s}-a(b,\bar{s})}{b-a(b,\bar{s})}$

is the resulting maximal feasible probability of trade. The two black straight lines represent $x \mapsto s - \rho(b_1, \bar{s})(x - b_1)$ and $x \mapsto s - \rho(b_2, \bar{s})(x - b_1)$, respectively. The line $x \mapsto s - \rho(b_1, \bar{s})(x - b_1)$ intersects the horizontal line of $\underline{s}$ at $b_1 + \frac{\bar{s}-\underline{s}}{\rho(b_1,\bar{s})}$, and the line $x \mapsto s - \rho(b_2, \bar{s})(x - b_1)$ intersects at $b_2 + \frac{\bar{s}-\underline{s}}{\rho(b_2,\bar{s})}$. The red interval describes $[b_1 + \frac{\bar{s}-\underline{s}}{\rho(b_1,\bar{s})},\ b_2 + \frac{\bar{s}-\underline{s}}{\rho(b_2,\bar{s})}]$. Claim C.2 states that if $v(\,\cdot\,, \bar{s})$ is increasing over the blue interval, so is $v(\,\cdot\,, \underline{s})$ over the red one. The proof of Claim C.2 is very involved. We devote Appendix D to its proof.

From the graph, it is also easy to observe that the following two equations hold for all b: $a(b + \frac{\bar{s}-\underline{s}}{\rho(b,\bar{s})}, \underline{s}) = a(b, \bar{s})$ and $\rho(b + \frac{\bar{s}-\underline{s}}{\rho(b,\bar{s})}, \underline{s}) = \rho(b, \bar{s})$. Moreover, by Claim B.2, we have $a^*(s) = a(b^*(s), s)$ and $\rho^*(s) = \rho(b^*(s), s)$. We will use these relationships in the proof of part (ii) of Proposition 3.

Proof of part (ii) of Proposition 3. Because $v(\,\cdot\,, \bar{s})$ is strictly concave by Claim B.3 and achieves its maximum at $b^*(\bar{s})$, it is increasing over $[\mu - \bar{s},\ b^*(\bar{s})]$. By Claim C.2, we know that $v(\,\cdot\,, \underline{s})$ is increasing over $[\mu - \underline{s},\ b^*(\bar{s}) + \frac{\bar{s}-\underline{s}}{\rho^*(\bar{s})}]$. Therefore, $b^*(\underline{s}) \geq b^*(\bar{s}) + \frac{\bar{s}-\underline{s}}{\rho^*(\bar{s})} > b^*(\bar{s})$. Moreover, $a^*(\underline{s}) = a(b^*(\underline{s}), \underline{s}) \geq a(b^*(\bar{s}) + \frac{\bar{s}-\underline{s}}{\rho^*(\bar{s})}, \underline{s}) = a(b^*(\bar{s}), \bar{s}) = a^*(\bar{s})$, where the inequality is because $a(\,\cdot\,, s)$ is increasing. Similarly, $\rho^*(\underline{s}) \leq \rho^*(\bar{s})$. $\square$

C.3 Proof of part (iii)



(a) Illustration of a, b and b_s constructed in the proof of $\lim_{s \downarrow 0} v^*(s) = 1$

(b) Illustration of $\rho^*(s) \geq 1 - F(b^*(s)-)$

Figure C.3: Proof of part (iii) of Proposition 3

Proof of part (iii) of Proposition 3. To show $\lim_{s\downarrow 0} v^*(s) = 1$, it suffices to show that $\lim_{s\downarrow 0} v^*(s) \geq 1-\varepsilon$ for any $\varepsilon > 0$. Fix an arbitrary $\varepsilon \in (0, 1-\mu)$. Let $b = 1 - \frac{\varepsilon}{2}$. Pick $0 < a < \mu$ such that $-\frac{\mu-a}{b-a}(x-b) \leq c_F(x)$ for all $x \in [a,b]$. Let $\rho = \frac{\mu-a}{b-a}$ and $b_s = b - \frac{s}{\rho}$ for $s > 0$. See panel (a) in Figure C.3 for illustrations of a and b, and the construction of b_s. The slope of the red line is $-\rho$. Let $v = 1-\varepsilon$. It is easy to see that there exists $s' > 0$ such that for all $s < s'$, we have $a \in [0, \mu - s)$, $b_s \geq \mu - s$ and $v \in [a, b_s)$. Thus, $c^s_{a,b_s,v}$ is well-defined when $s < s'$. It is feasible if and only if

$$h(x; a, b_s, v, s) = s - \rho(b_s - v)\log\frac{x-v}{b_s - v} \leq c_F(x),\ \forall x \in [b_s, 1]. \tag{C.1}$$

Because $h(x; a, b_0, v, 0) = -\rho(b-v)\log\frac{x-v}{b-v} < c_F(x)$ for all $x \in [b, 1]$, uniform continuity implies that there exists $s'' > 0$ such that (C.1) holds for $s < s''$. Therefore, when $s < \min\{s', s''\}$, $c^s_{a,b_s,v}$ is feasible. This, in turn, implies $v^*(s) \geq v = 1-\varepsilon$ when $s < \min\{s', s''\}$. Hence, $\lim_{s\downarrow 0} v^*(s) \geq 1 - \varepsilon$. Because $v^*(s) < b^*(s) < 1$ for all s, $\lim_{s\downarrow 0} b^*(s) = 1$ too.

Consider $\lim_{s\downarrow 0} a^*(s)$ and $\lim_{s\downarrow 0} \rho^*(s)$ now. By Claim B.2, we know $a^*(s) = a(b^*(s), s)$. It is easy to see from panel (b) in Figure C.3 that $a^*(s) \leq \mathbb{E}[q|q < b^*(s)]$ and hence $\rho^*(s) \geq 1 - F(b^*(s)-)$. This implies $\lim_{s\downarrow 0} \rho^*(s) \geq \lim_{s\downarrow 0} 1 - F(b^*(s)-) = 1 - F(1-)$. On the other hand, because $c^s_{a^*(s),b^*(s),v^*(s)}$ is feasible, we know $s - \rho^*(s)(x - b^*(s)) \leq c_F(x)$ for all $x \in [a^*(s), b^*(s)]$. For any $\mu < x < 1$, $x \in [a^*(s), b^*(s)]$ when s is sufficiently small. Hence, for sufficiently small s, $\rho^*(s) \leq \frac{c_F(x)-s}{b^*(s)-x}$, implying $\lim_{s\downarrow 0} \rho^*(s) \leq \frac{c_F(x)}{1-x} = \frac{c_F(x)-c_F(1)}{1-x}$. Letting $x \uparrow 1$, we obtain $\lim_{s\downarrow 0} \rho^*(s) \leq -c'_F(1-) = 1 - F(1-)$. Therefore, $\lim_{s\downarrow 0} \rho^*(s) = 1 - F(1-)$. Because $\rho^*(s) = \frac{\mu - s - a^*(s)}{b^*(s) - a^*(s)}$, we know $a^*(s) = \frac{\mu - s - \rho^*(s) b^*(s)}{1 - \rho^*(s)}$. Therefore, $\lim_{s\downarrow 0} a^*(s) = \frac{\mu - 1(1-F(1-))}{F(1-)} = \mathbb{E}[q|q < 1]$.

□

Appendix D Proof of Claim C.2

We prove Claim C.2 by contradiction. For this purpose, we maintain the assumption that Claim C.2 does not hold in this section.

D.1 A Special interval

The following claim shows the existence of a special interval $[b_1, b_2]$ that satisfies several properties. We will focus exclusively on this interval throughout the rest of the proof.

Claim D.1. *There exists $b_1 < b_2$ such that the following properties hold:*

(i) $v(b, \bar{s})$ is strictly increasing over $[b_1, b_2]$, but $v(b, \underline{s})$ is strictly decreasing over $[b_1 + \frac{\bar{s}-\underline{s}}{\rho(b_1,\bar{s})},\ b_2 + \frac{\bar{s}-\underline{s}}{\rho(b_2,\bar{s})}]$.

(ii) $h(x; a(b_1, \bar{s}), b_1, v(b_1, \bar{s}), \bar{s}) > \underline{s}$ at $x = b_2 + \frac{\bar{s}-\underline{s}}{\rho(b_2,\bar{s})}$.

Claim D.1 can be best understood from Figure D.1. Similarly as in Figure C.2, the blue line represents the desired interval $[b_1, b_2]$ for $\bar{s}$ and the red line represents the interval $[b_1 + \frac{\bar{s}-\underline{s}}{\rho(b_1,\bar{s})},\ b_2 + \frac{\bar{s}-\underline{s}}{\rho(b_2,\bar{s})}]$ for $\underline{s}$. Part (i) requires that $v(b, \bar{s})$ be increasing over the blue range and $v(b, \underline{s})$ is decreasing over the red range. The green curve in Figure D.1 represents $c^{\bar{s}}_{a(b_1,\bar{s}),b_1,v(b_1,\bar{s})}$. Part (ii) requires that this curve at $x = b_2 + \frac{\bar{s}-\underline{s}}{\rho(b_1,\bar{s})}$ be above the horizontal line of $\underline{s}$. In a nutshell, part (i) is a direct consequence of our assumption that Claim C.2 does not hold. Part (ii) then comes from continuity. It basically requires that the interval $[b_1, b_2]$ be small.

Proof. For ease of exposition, define $\phi(b) \equiv b + \frac{\bar{s}-\underline{s}}{\rho(b,\bar{s})}$. Because $\rho(b, \bar{s})$ is continuous in b, so is ϕ. Because Claim C.2 does not hold, there exists an interval $[b_1', b_2']$ such that $v(b, \bar{s})$ is strictly increasing over it, but $v(b, \underline{s})$ is not strictly increasing over $[\phi(b_1'), \phi(b_2')]$. Because $v(b, \underline{s})$ is strictly concave in b by Claim B.3, there exists $b_1 \in (b_1', b_2')$ such that $v(b, \underline{s})$ is strictly decreasing over $[\phi(b_1), \phi(b_2')]$. Hence, the

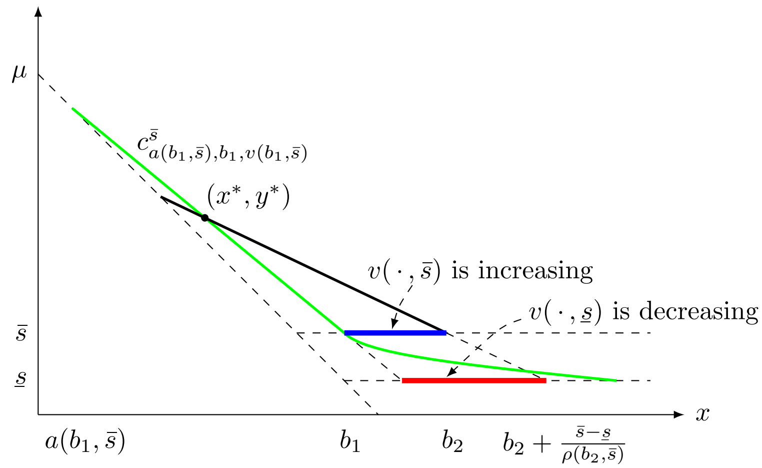


Figure D.1: Graph for Claim D.1

interval $[b_1, b_2']$ satisfies part (i). To satisfy part (ii), we need to shrink it to $[b_1, b_2]$ for some b_2 close enough to b_1. Clearly, shrinking $[b_1, b_2']$ does not violate part (i).

Because $h(\phi(b_1); a(b_1,\bar{s}), b_1, v(b_1,\bar{s}), \bar{s}) > \underline{s}$ by construction of h function, there exists, by continuity, $b_2 \in (b_1, b_2']$ such that $h(\phi(b_2); a(b_1,\bar{s}), b_1, v(b_1,\bar{s}), \bar{s}) > \underline{s}$. The interval $[b_1, b_2]$ is desired. $\square$

D.2 Change of variables

In what follows, we will derive a contradiction to Claim D.1. The analysis is complicated and it is not convenient to work with signal cutoff b. The major reason is that the domain of the signal cutoffs are different for different search costs. For instance, the domain is $[b_1, b_2]$ for $s = \underline{s}$, whereas it becomes $[b_1 + \frac{\bar{s}-\underline{s}}{\rho(b_1,\bar{s})},\ b_2 + \frac{\bar{s}-\underline{s}}{\rho(b_2,\bar{s})}]$ for $s = \bar{s}$. For more convenient analysis, we will work with the probability of trade instead of signal cutoff, as the situation we focus on has a very natural one-to-one mapping between the probabilities of trade and signal cutoffs.

Figure D.2, which basically reproduces Figure D.1, gives an explanation of this one-to-one mapping and the change of variables. The two blue curves represent $c^{\bar{s}}_{a(b_1,\bar{s}),b_1,v(b_1,\bar{s})}$ and $c^{\bar{s}}_{a(b_2,\bar{s}),b_2,v(b_2,\bar{s})}$, respectively as before. Similarly as the situation in the proof of Claim B.3, these two curves intersect at some point above $\bar{s}$. See the black dot in Figure D.2. Denote this intersection by (x^*, y^*). Let $\underline{\rho} \equiv \rho(b_2, \bar{s})$ and $\bar{\rho} \equiv \rho(b_1, \bar{s})$. We know $\underline{\rho} < \bar{\rho}$. Every $b \in [b_1, b_2]$ is uniquely identified by a $\rho \in [\underline{\rho}, \bar{\rho}]$ according to $b = x^* + \frac{y^* - \bar{s}}{\rho}$. This is illustrated by the red line that starts at (x^*, y^*) and has slope $-\rho$. The intersection of this line and the horizontal $\bar{s}$ line is exactly $b = x^* + \frac{y^* - \bar{s}}{\rho}$. Obviously, for $\underline{s}$, a similar relationship between signal cutoff and probability of trade exists.

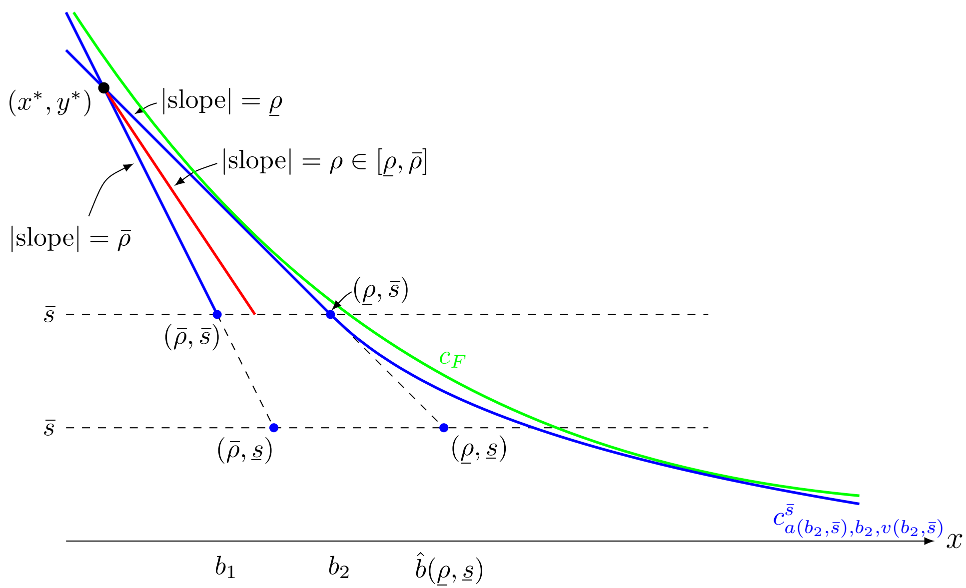


Figure D.2: Change of variables

Formally, for $(\rho, s) \in [\underline{\rho}, \bar{\rho}] \times [\underline{s}, \bar{s}]$, define $\hat{b}(\rho, s) \equiv x^* + \frac{y^* - s}{\rho}$. Note that $\hat{b}(\rho, s)$ is decreasing in ρ. Moreover, we have $b_1 = \hat{b}(\bar{\rho}, \bar{s})$, $b_2 = \hat{b}(\underline{\rho}, \bar{s})$, $b_1 + \frac{\bar{s} - \underline{s}}{\rho(b_1, \bar{s})} = \hat{b}(\bar{\rho}, \underline{s})$, and $b_2 + \frac{\bar{s} - \underline{s}}{\rho(b_2, \bar{s})} = \hat{b}(\underline{\rho}, \underline{s})$. This is also illustrated in Figure D.2. We label the four blue dots by the corresponding pair $(\rho, s) \in \{\underline{\rho}, \bar{\rho}\} \times \{\underline{s}, \bar{s}\}$. This change of variables effectively eliminates the problem of moving domains of the signal cutoffs for different search costs.

Similarly, for $(\rho, s) \in \{\underline{\rho}, \bar{\rho}\} \times \{\underline{s}, \bar{s}\}$, let $\hat{v}(\rho, s) \equiv v(\hat{b}(\rho, s), s)$.[3] For instance, $\hat{v}(\bar{\rho}, \bar{s}) = v(b_1, \bar{s})$ is just the highest consumer surplus given signal cutoff b_1 under

[3] We can define $\hat{v}$ for all pairs $(\rho, s) \in [\underline{\rho}, \bar{\rho}] \times [\underline{s}, \bar{s}]$. But in the following analysis, we will never need such $\hat{v}(\rho, s)$ if $(\rho, s) \notin \{\underline{\rho}, \bar{\rho}\} \times \{\underline{s}, \bar{s}\}$.

search cost $\bar{s}$. By Claims C.1 and D.1, we obtain a chain of inequalities

$$\hat{v}(\bar{\rho}, \bar{s}) < \hat{v}(\underline{\rho}, \bar{s}) < \hat{v}(\underline{\rho}, \underline{s}) < \hat{v}(\bar{\rho}, \underline{s}), \tag{D.1}$$

where the first and the third inequalities come from part (i) of Claim D.1. The second inequality comes from Claim C.1. These relationships, which are summarized in Figure D.3, are important for the following analysis.

$$\begin{array}{ccc} & \text{CD.1} & \\ \hat{v}(\bar{\rho}, \bar{s}) & < & \hat{v}(\underline{\rho}, \bar{s}) \\ & & \wedge \ \ \text{CC.1} \\ \hat{v}(\bar{\rho}, \underline{s}) & > & \hat{v}(\underline{\rho}, \underline{s}) \\ & \text{CD.1} & \end{array}$$

Figure D.3: Comparison of $\hat{v}$ for different pairs of (ρ, s)

Finally, for $(\rho, s) \in [\underline{\rho}, \bar{\rho}] \times [\underline{s}, \bar{s}]$ and $v < \hat{b}(\rho, s)$, define function

$$\hat{h}(x; \rho, v, s) \equiv s - \rho(\hat{b}(\rho, s) - v) \log \frac{x - v}{\hat{b}(\rho, s) - v}, \ \forall x \in [\hat{b}(\rho, s), 1].$$

Function $\hat{h}$ is the change-of-variables analogue of function h. Instead of using signal cutoff b and low match value atom a as parameters, $\hat{h}$ directly specifies the probability of trade ρ and the corresponding signal cutoff by $\hat{b}(\rho, s)$. In particular, when $(\rho, s) \in \{\underline{\rho}, \bar{\rho}\} \times \{\underline{s}, \bar{s}\}$, for any v, $\hat{h}(\,\cdot\,; \rho, v, s)$ just coincides with $h(\,\cdot\,; a(\hat{b}(\rho, s), s), \hat{b}(\rho, s), v, s)$ — the h curve for signal cutoff $\hat{b}(\rho, s)$ with the maximal feasible probability of trade under search cost s.[4] Part (ii) of Claim D.1 then can be equivalently expressed as $\hat{h}(\hat{b}(\underline{\rho}, \underline{s}); \bar{\rho}, \hat{v}(\bar{\rho}, \bar{s}), \bar{s}) > \underline{s}$.

[4]We point out that when $\rho \neq \underline{\rho}$ or $\bar{\rho}$, the curve $\hat{h}(\,\cdot\,; \rho, v, s)$ may be different from the curve $h(\,\cdot\,; a(\hat{b}(\rho, s), s), \hat{b}(\rho, s), v, s)$, although we do not need this fact in the following analysis. This is because the maximal feasible probability of trade for signal cutoff $\hat{b}(\rho, s)$ under search cost s, i.e., $\rho(\hat{b}(\rho, s), s)$, may be strictly larger than ρ.

Two obvious properties of $\hat{h}$ are worth mentioning. First, it is smooth, not only in x, but in all parameters ρ, v, and s. It is this property that creates a useful "bridge" to link the highest consumer surplus $\hat{v}$ at the four discrete points $\{\underline{\rho}, \bar{\rho}\} \times \{\underline{s}, \bar{s}\}$.[5] Second, all else being equal, higher consumer surplus v leads to a higher $\hat{h}$ curve, i.e., $\hat{h}_v > 0$. This is simply an inherited property from h. See Claim A.1. We will use this property frequently below.

D.3 Binding feasibility constraints

For each pair $(\rho, s) \in \{\underline{\rho}, \bar{\rho}\} \times \{\underline{s}, \bar{s}\}$, $\hat{v}(\rho, s)$ is the corresponding highest feasible consumer surplus. It is straightforward to see, by uniform continuity, that some of the feasibility constraints $\hat{h}(x; \rho, \hat{v}(\rho, s), s) \leq c_F(x)$ for $x \in [\hat{b}(\rho, s), 1]$ must be binding. Among these four pairs, we are particularly interested in the binding constraints of $(\bar{\rho}, \bar{s})$ and $(\underline{\rho}, \underline{s})$.

Claim D.2. *Consider the pair $(\bar{\rho}, \bar{s})$. At the highest consumer surplus $\hat{v}(\bar{\rho}, \bar{s})$, there exists $x_\dagger \in (\hat{b}(\underline{\rho}, \bar{s}), 1]$ such that the feasibility constraint is binding at $x_\dagger$. That is, $\hat{h}(x_\dagger; \bar{\rho}, \hat{v}(\bar{\rho}, \bar{s}), \bar{s}) = c_F(x_\dagger) < \bar{s}$.*

There may be multiple binding points. We pick an arbitrary one, $x_\dagger$, and fix it throughout the following analysis. We do want to emphasize that this binding point must satisfy $x_\dagger > \hat{b}(\underline{\rho}, \bar{s}) = b_2$. In Figure D.2, this means $x_\dagger$ must appear to the right of b_2. This is obvious from the graph. The curve $h(\,\cdot\,; \bar{\rho}, \hat{v}(\bar{\rho}, \bar{s}), \bar{s})$ (not drawn) is below $\bar{s}$ by construction, whereas the green curve, which represents c_F, is above $\bar{s}$ over $[b_1, b_2]$.[6] Hence, binding cannot occur over this range. Consequently, $\hat{b}(\rho, \bar{s}) < x_\dagger$ for all $\rho \in [\underline{\rho}, \bar{\rho}]$. We will use this fact later.

[5] In contrast, for example, the function $h(x; a(b, s), b, v, s)$ may not be differentiable in b, as the lowest feasible atom $a(b, s)$ may not be differentiable.

[6] The fundamental reason is the feasibility of $c^{\bar{s}}_{a(b_2, \bar{s}), b_2, \hat{v}(b_2, \bar{s})}$. That is, the blue curve on the right is below c_F.

Claim D.3. *Consider the pair $(\underline{\rho}, \underline{s})$. At the highest consumer surplus $\hat{v}(\underline{\rho}, \underline{s})$, there exists $x^\dagger \in (\hat{b}(\underline{\rho}, \underline{s}),\, 1]$ such that the feasibility constraint is binding at $x^\dagger$. That is, $\hat{h}(x^\dagger; \underline{\rho}, \hat{v}(\underline{\rho}, \underline{s}), \underline{s}) = c_F(x^\dagger) < \underline{s}$.*

Similarly as the pair $(\bar{\rho}, \bar{s})$, there may be multiple binding points too for the pair $(\underline{\rho}, \underline{s})$. We pick an arbitrary one, $x^\dagger$, and fix it throughout the following analysis. Note that $\hat{b}(\underline{\rho}, \bar{s}) < x^\dagger$ implies $\hat{b}(\rho, s) < x^\dagger$ for all $(\rho, s) \in [\underline{\rho}, \bar{\rho}] \times [\underline{s}, \bar{s}]$. We will also use this fact later.

The following claim compares $x_\dagger$ from Claim D.2 and $x^\dagger$ from Claim D.3. This is where we use part (ii) of Claim D.1.

Claim D.4. *We have $x_\dagger \leq x^\dagger$.*

Proof. We compare the two curves, $\hat{h}(\,\cdot\,; \bar{\rho}, \hat{v}(\bar{\rho}, \bar{s}), \bar{s})$ and $\hat{h}(\,\cdot\,; \underline{\rho}, \hat{v}(\underline{\rho}, \underline{s}), \underline{s})$, over the interval $[\hat{b}(\underline{\rho}, \underline{s}),\, x^\dagger]$.

At $x = \hat{b}(\underline{\rho}, \underline{s})$, we know from part (ii) of Claim D.1 that

$$\hat{h}(\hat{b}(\underline{\rho}, \underline{s}); \bar{\rho}, \hat{v}(\bar{\rho}, \bar{s}), \bar{s}) > \underline{s} = \hat{h}(\hat{b}(\underline{\rho}, \underline{s}); \underline{\rho}, \hat{v}(\underline{\rho}, \underline{s}), \underline{s}). \tag{D.2}$$

At $x = x^\dagger$, we know

$$\hat{h}(x^\dagger; \bar{\rho}, \hat{v}(\bar{\rho}, \bar{s}), \bar{s}) \leq c_F(x^\dagger) = \hat{h}(x^\dagger; \underline{\rho}, \hat{v}(\underline{\rho}, \underline{s}), \underline{s}), \tag{D.3}$$

where the inequality comes from feasibility. Recall that $\hat{v}(\bar{\rho}, \bar{s}) < \hat{v}(\underline{\rho}, \underline{s})$ in (D.1). Then, combining (D.2) and (D.3), and applying a similar argument as that in the proof of Claim A.3, we can show that there exists $\tilde{x} \in (\hat{b}(\underline{\rho}, \underline{s})),\, x^\dagger)$ such that

$$\hat{h}_x(x; \bar{\rho}, \hat{v}(\bar{\rho}, \bar{s}), \bar{s}) < \hat{h}_x(x; \underline{\rho}, \hat{v}(\underline{\rho}, \underline{s}), \underline{s}),\ \forall x \geq \tilde{x}. \tag{D.4}$$

Combining (D.3) and (D.4) yields $\hat{h}(x; \bar{\rho}, \hat{v}(\bar{\rho}, \bar{s}), \bar{s}) < \hat{h}(x; \underline{\rho}, \hat{v}(\underline{\rho}, \underline{s}), \underline{s}) \leq c_F(x)$ for all

$x \in (x^\dagger, 1]$, where the second inequality is simply the feasibility constraint. Because $\hat{h}(x_\dagger; \bar{\rho}, \hat{v}(\bar{\rho}, \bar{s}), \bar{s}) = c_F(x_\dagger)$, we immediately know that $x_\dagger \leq x^\dagger$. $\square$

D.4 Consumer surplus that makes a certain feasibility constraint binding

Let $v^\dagger(\rho, s) : [\underline{\rho}, \bar{\rho}] \times [\underline{s}, \bar{s}] \to \mathbb{R}$ be the implicit function defined by equation

$$\hat{h}(x^\dagger; \rho, v^\dagger(\rho, s), s) = c_F(x^\dagger). \tag{D.5}$$

In words, $v^\dagger(\rho, s)$ is the value of the consumer surplus that makes the particular feasibility constraint $\hat{h}(x^\dagger; \rho, v^\dagger(\rho, s), s) \leq c_F(x^\dagger)$ binding. The following claim is straightforward. It verifies that $v^\dagger$ is well-defined and compares $v^\dagger$ and $\hat{v}$.

Claim D.5. *The following hold.*

(i) For every pair $(\rho, s) \in [\underline{\rho}, \bar{\rho}] \times [\underline{s}, \bar{s}]$, $v^\dagger(\rho, s) \in [0, \hat{b}(\rho, s))$ is well-defined.

(ii) For every pair $(\rho, s) \in \{\underline{\rho}, \bar{\rho}\} \times \{\underline{s}, \bar{s}\}$, we have $v^\dagger(\rho, s) \geq \hat{v}(\rho, s)$, with equality if $(\rho, s) = (\underline{\rho}, \underline{s})$.

Proof. Because $\hat{b}(\rho, s) < x^\dagger$ for all $(\rho, s) \in [\underline{\rho}, \bar{\rho}] \times [\underline{s}, \bar{s}]$ as we mentioned after Claim D.3, $x^\dagger$ is in the domain of every $\hat{h}(\,\cdot\,; \rho, v, s)$ curve. Applying the same argument as in the proof of Claim B.3 in the appendix, we know there exists $v \in [0, \hat{b}(\rho, s))$ such that $h(x^\dagger; \rho, v, s) \leq c_F(x^\dagger)$. Because $\lim_{v \uparrow \hat{b}(\rho,s)} h(x^\dagger; \rho, v, s) = s \geq \underline{s} > c_F(x^\dagger)$ and because $\hat{h}_v > 0$, we know there exists a unique $v^\dagger(\rho, s) \in [0, \hat{b}(\rho, s))$ such that (D.5) holds.

For $(\rho, s) \in \{\underline{\rho}, \bar{\rho}\} \times \{\underline{s}, \bar{s}\}$, we know $\hat{h}(x^\dagger; \rho, \hat{v}(\rho, s), s) \leq c_F(x^\dagger)$ by feasibility. Because $\hat{h}(x^\dagger; \rho, v^\dagger(\rho, s), s) = c_F(x^\dagger)$ by construction and because $\hat{h}_v > 0$ again, we must have $v^\dagger(\rho, s) \geq \hat{v}(\rho, s)$. Clearly, we have $v^\dagger(\underline{\rho}, \underline{s}) = \hat{v}(\underline{\rho}, \underline{s})$ from the definition of $x^\dagger$. $\square$

The greatest advantage of $v^\dagger$ is its differentiability by the implicit function theorem. The following claim, making use of this property, compares $v^\dagger(\underline{\rho}, \bar{s})$ and $v^\dagger(\bar{\rho}, \bar{s})$.

Claim D.6. *We have $v^\dagger(\underline{\rho}, \bar{s}) < v^\dagger(\bar{\rho}, \bar{s})$.*

Proof. By the implicit function theorem, we can calculate

$$
\begin{aligned}
v^\dagger_{\rho s} =& \frac{(x^\dagger - v^\dagger)(x^\dagger - x^*)\left(x^\dagger - \hat{b} - (\hat{b} - v^\dagger)\log\frac{x^\dagger - v^\dagger}{\hat{b} - v^\dagger}\right)\log\frac{x^\dagger - v^\dagger}{\hat{b} - v^\dagger}}{\rho^2\left(\hat{b} - x^\dagger + (x^\dagger - v^\dagger)\log\frac{x^\dagger - v^\dagger}{\hat{b} - v^\dagger}\right)^3} \\
&+ \frac{(x^\dagger - v^\dagger)\log\frac{x^\dagger - v^\dagger}{\hat{b} - v^\dagger}}{\rho^2\left(\hat{b} - x^\dagger + (x^\dagger - v^\dagger)\log\frac{x^\dagger - v^\dagger}{\hat{b} - v^\dagger}\right)},
\end{aligned}
$$

where $\hat{b}$ is $\hat{b}(\rho, s)$ and $v^\dagger$ is $v^\dagger(\rho, s)$ for short. Recall that $x^\dagger > \hat{b}(\rho, s)$ for all (ρ, s), as we mentioned after Claim D.3. This directly implies $x^\dagger > x^*$ as $x^* < \hat{b}(\bar{\rho}, \bar{s})$. Because $\hat{b}(\rho, s) > v^\dagger(\rho, s)$ for all (ρ, s), we also have $x^\dagger > v^\dagger(\rho, s)$ for all (ρ, s). Because $x^\dagger - \hat{b} - (\hat{b} - v^\dagger)\log\frac{x^\dagger - v^\dagger}{\hat{b} - v^\dagger} > x^\dagger - \hat{b} - (\hat{b} - v^\dagger)\left(\frac{x^\dagger - v^\dagger}{\hat{b} - v^\dagger} - 1\right) = 0$ and $\hat{b} - x^\dagger + (x^\dagger - v^\dagger)\log\frac{x^\dagger - v^\dagger}{\hat{b} - v^\dagger} > \hat{b} - x^\dagger - (x^\dagger - v^\dagger)\left(\frac{\hat{b} - v^\dagger}{x^\dagger - v^\dagger} - 1\right) = 0$, we know $v^\dagger_{s\rho} > 0$. Therefore, $v^\dagger(\bar{\rho}, \bar{s}) - v^\dagger(\underline{\rho}, \bar{s}) > v^\dagger(\bar{\rho}, \underline{s}) - v^\dagger(\underline{\rho}, \underline{s}) \geq \hat{v}(\bar{\rho}, \underline{s}) - \hat{v}(\underline{\rho}, \underline{s}) > 0$, where the second inequality comes from $v^\dagger(\bar{\rho}, \underline{s}) \geq \hat{v}(\bar{\rho}, \underline{s})$ and $v^\dagger(\underline{\rho}, \underline{s}) = \hat{v}(\underline{\rho}, \underline{s})$. The last inequality comes from (D.1). Therefore, $v^\dagger(\underline{\rho}, \bar{s}) < v^\dagger(\bar{\rho}, \bar{s})$. □

By Claim D.4, we know $x_\dagger \leq x^\dagger$. But the fact $\hat{v}(\bar{\rho}, \bar{s}) < \hat{v}(\underline{\rho}, \bar{s})$ from part (i) of Claim D.1 and Claim D.6 directly rule out the possibility $x_\dagger = x^\dagger$. This is stated in the next claim. It also compares the two curves, $\hat{h}(\,\cdot\,; \bar{\rho}, v^\dagger(\bar{\rho}, \bar{s}), \bar{s})$ and $\hat{h}(\,\cdot\,; \underline{\rho}, v^\dagger(\underline{\rho}, \bar{s}), \bar{s})$, at $x = x_\dagger$. This comparison is a direct corollary of Claim D.6 and the single-crossing property established in Claim A.3.

Claim D.7. *We have $x_\dagger < x^\dagger$ and*

$$
\hat{h}(x_\dagger; \bar{\rho}, v^\dagger(\bar{\rho}, \bar{s}), \bar{s}) < \hat{h}(x_\dagger; \underline{\rho}, v^\dagger(\underline{\rho}, \bar{s}), \bar{s}). \tag{D.6}
$$

Proof. If $x_\dagger = x^\dagger$, then $v_\dagger(\bar{\rho}, \bar{s}) = v^\dagger(\bar{\rho}, \bar{s})$. Hence, $\hat{v}(\bar{\rho}, \bar{s}) = v^\dagger(\bar{\rho}, \bar{s})$. Claim D.6 then implies $\hat{v}(\bar{\rho}, \bar{s}) > v^\dagger(\underline{\rho}, \bar{s}) \geq \hat{v}(\underline{\rho}, \bar{s})$, contradicting the fact $\hat{v}(\bar{\rho}, \bar{s}) < \hat{v}(\underline{\rho}, \bar{s})$. Therefore, by Claim D.4, we must have $x_\dagger < x^\dagger$.

Recall $\hat{h}(\,\cdot\,; \bar{\rho}, v^\dagger(\bar{\rho}, \bar{s}), \bar{s}) = h(\,\cdot\,; a(b_1, \bar{s}), b_1, v^\dagger(\bar{\rho}, \bar{s}), \bar{s})$ and $\hat{h}(\,\cdot\,; \underline{\rho}, v^\dagger(\underline{\rho}, \bar{s}), \bar{s}) = h(\,\cdot\,; a(b_2, \underline{s}), b_2, v^\dagger(\underline{\rho}, \bar{s}), \bar{s})$. Because (i) $b_1 < b_2$, (ii) $v^\dagger(\bar{\rho}, \bar{s}) > v^\dagger(\underline{\rho}, \bar{s})$ by Claim D.6, (iii) these two curves intersect at $x^\dagger$ by the definition of $v^\dagger(\bar{\rho}, \bar{s})$ and $v^\dagger(\underline{\rho}, \bar{s})$, and (iv) $x_\dagger < x^\dagger$, Claim A.3 immediately implies (D.6). $\square$

Similarly as $v^\dagger$, for $\rho \in \{\bar{\rho}, \underline{\rho}\}$, let $v_\dagger(\rho, \bar{s}) \in [0, \hat{b}(\rho, s))$ be the value of the consumer surplus that makes the feasibility constraint $\hat{h}(x_\dagger; \rho, v_\dagger(\rho, \bar{s}), \bar{s}) \leq c_F(x_\dagger)$ binding. That is,

$$\hat{h}(x_\dagger; \rho, v_\dagger(\rho, \bar{s}), \bar{s}) = c_F(x_\dagger). \tag{D.7}$$

Clearly, $v_\dagger(\bar{\rho}, \bar{s}) = \hat{v}(\bar{\rho}, \bar{s})$ by the definition of $x_\dagger$ in Claim D.2. As for Claim D.5, we can similarly show that $v_\dagger(\underline{\rho}, \bar{s})$ is well-defined and $v_\dagger(\underline{\rho}, \bar{s}) \geq \hat{v}(\underline{\rho}, \bar{s})$.[7]

The next claim compares $v_\dagger(\underline{\rho}, \bar{s})$ and $v^\dagger(\underline{\rho}, \bar{s})$.

Claim D.8. *We have $v_\dagger(\underline{\rho}, \bar{s}) < v^\dagger(\underline{\rho}, \bar{s})$.*

Proof. Notice that

$$\hat{h}(x_\dagger; \underline{\rho}, v_\dagger(\underline{\rho}, \bar{s}), \bar{s}) = \hat{h}(x_\dagger; \bar{\rho}, v_\dagger(\bar{\rho}, \bar{s}), \bar{s}) \leq \hat{h}(x_\dagger; \bar{\rho}, v^\dagger(\bar{\rho}, \bar{s}), \bar{s}) < \hat{h}(x_\dagger; \underline{\rho}, v^\dagger(\underline{\rho}, \bar{s}), \bar{s}),$$

where the equality comes from the definition of $v_\dagger$, i.e., (D.7). The first inequality comes from $v_\dagger(\bar{\rho}, \bar{s}) = \hat{v}(\bar{\rho}, \bar{s}) \leq v^\dagger(\bar{\rho}, \bar{s})$ and $\hat{h}_v > 0$. The last inequality comes from Claim D.7. Hence, because $\hat{h}_v > 0$ again, we know $v_\dagger(\underline{\rho}, \bar{s}) < v^\dagger(\bar{\rho}, \bar{s})$. $\square$

Figure D.4 provides an illustration of the $\hat{h}$ curves involved in Claims D.6 to D.8. The green curve represents c_F. The solid red one represents $\hat{h}(\,\cdot\,; \bar{\rho}, v^\dagger(\bar{\rho}, \bar{s}), \bar{s})$ and the solid blue one represents $\hat{h}(\,\cdot\,; \underline{\rho}, v^\dagger(\underline{\rho}, \bar{s}), \bar{s})$. By the definition of $v^\dagger$, these three curves

[7] For any $\rho \in (\underline{\rho}, \bar{\rho})$, we can similarly define $v_\dagger(\rho, \bar{s})$. But we do not need it.

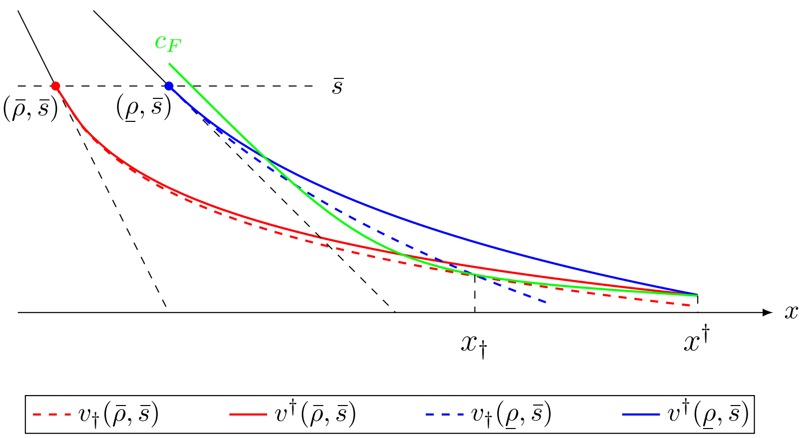


Figure D.4: Illustration of Claims D.6 to D.8

intersect at $x^\dagger$. By Claim D.6 and the single-crossing property established in Claim A.3, this red curve is, to the left of $x^\dagger$, everywhere below the blue one. The dashed red curve is $\hat{h}(\,\cdot\,; \bar{\rho}, v_\dagger(\underline{\rho}, \bar{s}), \bar{s})$, or equivalently $\hat{h}(\,\cdot\,; \bar{\rho}, \hat{v}(\underline{\rho}, \bar{s}), \bar{s})$. It is the highest feasible curve (i.e., everywhere below c_F) for $(\bar{\rho}, \bar{s})$, and intersects c_F at $x_\dagger$, which is to the left of $x^\dagger$ by Claim D.7. It is everywhere below the solid red curve, simply because it is below c_F at $x^\dagger$ by feasibility while the solid red curve intersects. Hence, the solid blue curve is above c_F at $x_\dagger$, which in turn implies that it is above the dashed blue curve at $x_\dagger$, which presents $\hat{h}(\,\cdot\,; \underline{\rho}, v_\dagger(\underline{\rho}, \bar{s}), \bar{s})$ and intersects c_F at $x_\dagger$. This directly implies $v_\dagger(\underline{\rho}, \bar{s}) < v^\dagger(\underline{\rho}, \bar{s})$, as is stated in Claim D.8.

Based on the above analysis, Claim D.10 below will establish the last comparison, $v_\dagger(\underline{\rho}, \bar{s}) \leq v_\dagger(\bar{\rho}, \bar{s})$, from which we can derive a contradiction. But to prove this inequality, we need an additional result: $x^* \leq v_\dagger(\bar{\rho}, \bar{s})$. It is a direct consequence of Claim D.1. Recall that x^* is the intersection of the two blue curves in Figure D.2.

Claim D.9. *We have $x^* \leq \hat{v}(\bar{\rho}, \bar{s})$.*

Proof. Suppose, by contradiction, $x^* > \hat{v}(\bar{\rho}, \bar{s})$. Pick $\lambda \in (0, 1)$ close enough to 1 such that $x^* > v^\lambda \equiv \lambda \hat{v}(\bar{\rho}, \bar{s}) + (1 - \lambda)\hat{v}(\underline{\rho}, \bar{s})$. Let $\rho^\lambda \in (\underline{\rho}, \bar{\rho})$ be the solution to $\hat{b}(\rho^\lambda, \bar{s}) = \lambda \hat{b}(\bar{\rho}, \bar{s}) + (1 - \lambda)\hat{b}(\underline{\rho}, \bar{s})$. Note that $\hat{b}(\rho^\lambda, \bar{s}) > v^\lambda$, as $\hat{b}(\rho^\lambda, \bar{s}) > \hat{b}(\bar{\rho}, \bar{s}) > x^*$.

Inequality (B.4) shows, in terms of the current $\hat{h}$ function, that $\hat{h}(x; \rho^\lambda, v^\lambda, \bar{s}) \leq$

$c_F(x)$ for all $x \in [\hat{b}(\rho^\lambda, \bar{s}),\, 1]$. For all $x \in [\hat{b}(\rho^\lambda, \bar{s}),\, 1]$ and $\rho \in [\rho^\lambda, \bar{\rho}]$, we have

$$\hat{h}_\rho(x; \rho, v^\lambda, \bar{s}) = -\frac{1}{\rho}\left[(y^* - \bar{s}) + \rho(x^* - v^\lambda)\log\frac{x - v^\lambda}{\hat{b}(\rho, \bar{s}) - v^\lambda}\right] < 0,$$

where the inequality comes from our assumption that $x^* > v^\lambda$. Because $\rho^\lambda < \bar{\rho}$, we then have $\hat{h}(x; \bar{\rho}, v^\lambda, \bar{s}) < \hat{h}(x; \rho^\lambda, v^\lambda, \bar{s}) \leq c_F(x)$ for all $x \in [\hat{b}(\rho^\lambda, \bar{s}),\, 1]$. This in turn implies $\hat{v}(\bar{\rho}, \bar{s}) \geq v^\lambda$, because $\hat{v}(\bar{\rho}, \bar{s})$ by definition is the highest consumer surplus for the pair $(\bar{\rho}, \bar{s})$. But $v^\lambda = \lambda\hat{v}(\bar{\rho}, \bar{s}) + (1-\lambda)\hat{v}(\underline{\rho}, \bar{s}) > \hat{v}(\bar{\rho}, \bar{s})$, because $\hat{v}(\bar{\rho}, \bar{s}) < \hat{v}(\underline{\rho}, \bar{s})$ by Claim D.1. This is a contradiction. Therefore, we must have $x^* \leq \hat{v}(\bar{\rho}, \bar{s})$. □

We are now ready to prove the last inequality.

Claim D.10. *We have* $v_\dagger(\underline{\rho}, \bar{s}) \leq v_\dagger(\bar{\rho}, \bar{s})$.

Proof. Suppose, by contradiction, $v_\dagger(\bar{\rho}, \bar{s}) < v_\dagger(\underline{\rho}, \bar{s})$. Recall that $v_\dagger(\bar{\rho}, \bar{s}) = \hat{v}(\bar{\rho}, \bar{s}) = v(b_1, \bar{s})$. Combining Claims D.6, D.8, and D.9, we obtain a chain of inequalities

$$x^* \leq v_\dagger(\bar{\rho}, \bar{s}) < v_\dagger(\underline{\rho}, \bar{s}) < v^\dagger(\underline{\rho}, \bar{s}) < v^\dagger(\bar{\rho}, \bar{s}). \tag{D.8}$$

Because $\hat{h}_v > 0$, we know

$$\begin{aligned}\hat{h}(x_\dagger; \bar{\rho}, v^\dagger(\bar{\rho}, \bar{s}), \bar{s}) =& \hat{h}(x_\dagger; \bar{\rho}, v^\dagger(\bar{\rho}, \bar{s}), \bar{s}) - \hat{h}(x_\dagger; \bar{\rho}, v_\dagger(\bar{\rho}, \bar{s}), \bar{s}) + c_F(x_\dagger) \\ >& \hat{h}(x_\dagger; \bar{\rho}, v^\dagger(\underline{\rho}, \bar{s}), \bar{s}) - \hat{h}(x_\dagger; \bar{\rho}, v_\dagger(\underline{\rho}, \bar{s}), \bar{s}) + c_F(x_\dagger).\end{aligned} \tag{D.9}$$

Note that $\hat{h}_{\rho v}(x_\dagger; \rho, v, \bar{s}) = \frac{(v - x^*)(x_\dagger - \hat{b}(\rho, \bar{s}))}{(x_\dagger - v)(\hat{b}(\rho, \bar{s}) - v)} + \log\frac{x_\dagger - v}{\hat{b}(\rho, \bar{s}) - v}$. Thus, for all $\rho \in [\underline{\rho}, \bar{\rho}]$ and $v \in [v_\dagger(\underline{\rho}, \bar{s}),\; v^\dagger(\underline{\rho}, \bar{s})]$, we have $\hat{h}_{\rho v} > 0$.[8] Therefore,

$$\begin{aligned}\hat{h}(x_\dagger; \bar{\rho}, v^\dagger(\underline{\rho}, \bar{s}), \bar{s}) - \hat{h}(x_\dagger; \bar{\rho}, v_\dagger(\underline{\rho}, \bar{s}), \bar{s}) &> \hat{h}(x_\dagger; \underline{\rho}, v^\dagger(\underline{\rho}, \bar{s}), \bar{s}) - \hat{h}(x_\dagger; \underline{\rho}, v_\dagger(\underline{\rho}, \bar{s}), \bar{s}) \\ &= \hat{h}(x_\dagger; \underline{\rho}, v^\dagger(\underline{\rho}, \bar{s}), \bar{s}) - c_F(x_\dagger).\end{aligned} \tag{D.10}$$

[8] For every $v \in [v_\dagger(\underline{\rho}, \bar{s}),\; v^\dagger(\underline{\rho}, \bar{s})]$, (D.8) implies $x^* < v < v^\dagger(\bar{\rho}, \bar{s}) < \hat{b}(\bar{\rho}, \bar{s}) \leq \hat{b}(\rho, \bar{s}) < x_\dagger$ for all $\rho \in [\underline{\rho}, \bar{\rho}]$.

Combining (D.9) and (D.10), we obtain $\hat{h}(x_\dagger; \bar{\rho}, v^\dagger(\bar{\rho}, \bar{s}), \bar{s}) > \hat{h}(x_\dagger; \underline{\rho}, v^\dagger(\underline{\rho}, \bar{s}), \bar{s})$, which contradicts (D.6). See Figure D.4. The above inequality means that the solid red curve is higher than the solid blue one at $x_\dagger$, which contradicts the graph. Therefore, we must have $v_\dagger(\underline{\rho}, \bar{s}) \leq v_\dagger(\bar{\rho}, \bar{s})$. □

D.5 Contradiction

We are ready to prove Claim C.2.

Proof of Claim C.2. From the definitions of $x_\dagger$ from Claim D.2 and $v_\dagger$ from (D.7), we know $v_\dagger(\bar{\rho}, \bar{s}) = \hat{v}(\bar{\rho}, \bar{s})$ and $v_\dagger(\underline{\rho}, \bar{s}) \geq \hat{v}(\underline{\rho}, \bar{s})$. Claim D.10 then implies $\hat{v}(\bar{\rho}, \bar{s}) \geq \hat{v}(\underline{\rho}, \bar{s})$. This contradicts the fact that $\hat{v}(\rho, \bar{s})$ is strictly decreasing is over $[\underline{\rho}, \bar{\rho}]$ (or equivalently, $v(b, \bar{s})$ is strictly increasing over $[b_1, b_2]$ by Claim D.1). This contradiction originates from our hypothesis that Claim C.2 does not hold. Therefore, Claim C.2 must hold, completing the proof. □

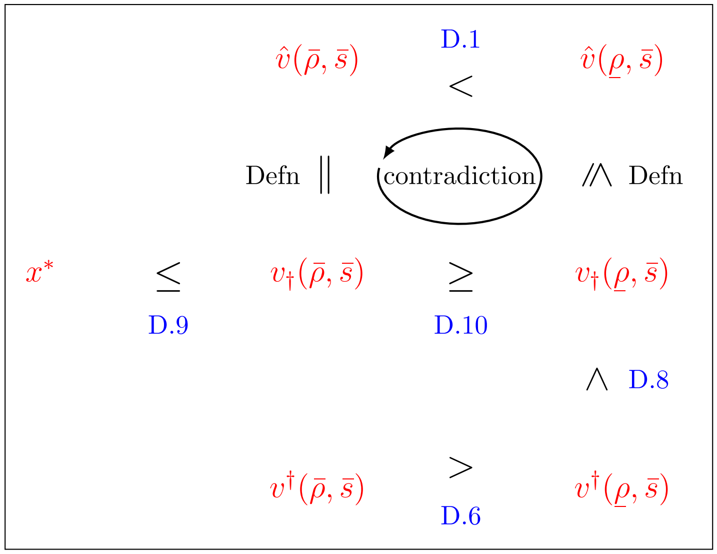
The diagram in panel (a) in Figure D.5 provides an overview of the proved inequalities. The diagram in panel (b) summarizes the above whole analysis.

Appendix E Proof of Proposition 4

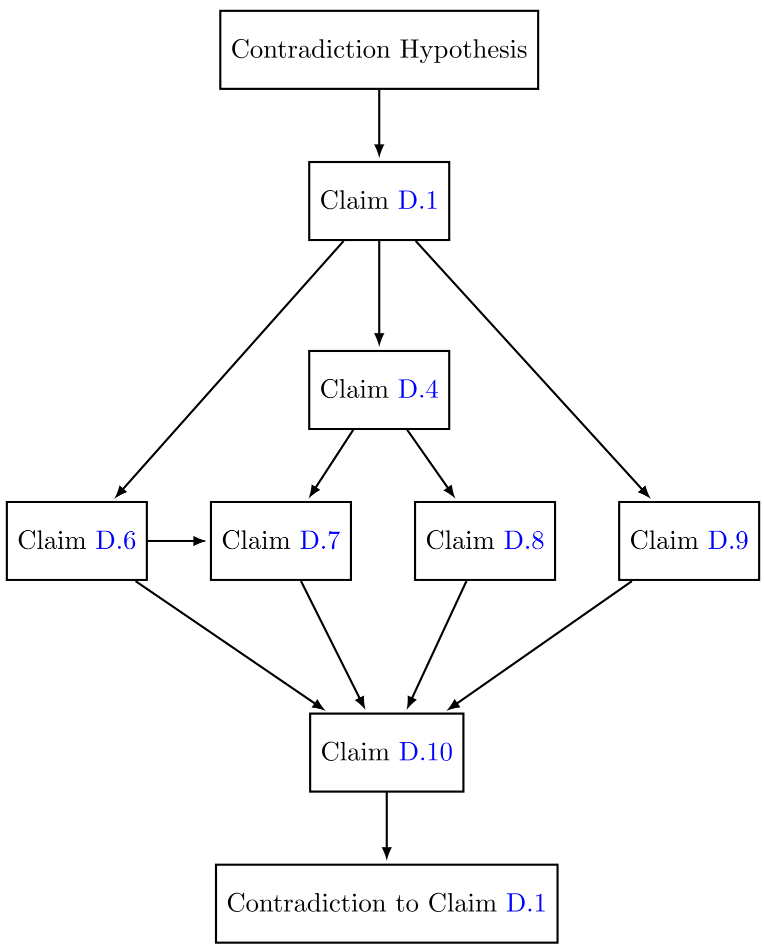
We first prove the desired result when G itself a unit-elastic demand signal distribution. This is Claim E.1. Then, we apply it to prove Proposition 4.

Claim E.1. *Suppose $c_G = c_{a,b,v}$ for some values a, b, and v. Assume $b > \mu - s$. If (7) holds, then there exist $b' \in [\mu - s, b)$ and $v' \in (v, b')$ such that $c_{a,b',v'} \leq c_{a,b,v}$.*

Proof. Let $\bar{x}$ be the intersection of $h(x; a, b, v)$ and the x-axis so that $h(\bar{x}; a, b, v) = 0$.



(a) Derived inequalities in the proof of Claim C.2



(b) Relationships between Claims D.1 to D.10

Figure D.5: Summary of the proof of Claim C.2

It is easy to calculate

$$
\begin{aligned}
h_b(\bar{x}; a, b, v) &= \frac{\mu - s - a}{(b-a)^2}\left[b - a - (v-a)\log\frac{\bar{x}-v}{b-v}\right] \\
&= \frac{\mu - s - a}{(b-a)^2}\left[b - a - (v-a)\frac{s(b-a)}{(\mu-s-a)(b-v)}\right] \\
&= \frac{\mu - s - a}{(b-a)}\left[1 - \frac{s(v-a)}{(\mu-s-a)(b-v)}\right] \\
&> 0,
\end{aligned}
$$

where the second equality comes from $h(\bar{x}; a, b, v) = 0$. The inequality is equivalent to (7) in the article. By continuity of h_b, $h(\bar{x}; a, \cdot, v)$ is strictly increasing over a neighborhood of b. This implies that there exists $b' \in [\mu - s,\, b)$ with $b' > v$ such that $h(\bar{x}; a, b', v) < 0$. By continuity of h, there exists $v' \in (v, b')$ such that $h(\bar{x}; a, b', v') \leq 0$. By Claim A.3, we know $h(x; a, b', v') \leq h(x; a, b, v)$ for all $x \in [b, \bar{x}]$, which in turn implies $c_{a,b',v'} \leq c_{a,b,v}$. □

Proof of Proposition 4. First, consider the case $v < a$. A careful examination of the proof of Proposition 1 reveals that $h(x; 0, \mu - s, v) < c_G(x)$ for all $x \in [\mu - s,\, 1]$. By uniform continuity, there exists $v' > v$ such that $h(x; 0, \mu - s, v') \leq c_G(x)$ for all $x \in [\mu - s,\, 1]$. Therefore, $c_{0,\mu-s,v'} \leq c_G$, implying that G is never consumer-optimal. Next, consider the case $a \leq v < b - \frac{b-a}{\mu-a}s$. By Proposition 1, we know $c_{a,b,v} \leq c_G$. By Claim E.1, we know that there exist $b' \in [\mu - s, b)$ and $v' \in (v, b')$ such that $c_{a,b',v'} \leq c_G$. Thus, G is never consumer-optimal.

Suppose now that $\pi \geq s$, or equivalently $\frac{\mu-s-a}{b-a}(b - v) \geq s$. We then know $\frac{\mu-a}{b-a}(b - v) > s$, which is equivalent to (7). □

Appendix F Consumers' mixed strategy

In this section, we explain why restricting attention to the consumers' strategies in which they purchase with probability one whenever they are indifferent is without loss of generality.

Formally, consider a signal distribution G. An equilibrium in which consumers also mix can be characterized by a triple (σ, η, v), where σ is the firms' mixed strategy over equilibrium signal cutoffs and v is the equilibrium consumer surplus, as in Section 5. The new component is function $\eta : [v, 1] \to [0, 1]$. It summarizes the consumers' mixed behavior in equilibrium when they are indifferent. For every $x \in [v, 1]$, $\eta(x)$ is the probability that consumers purchase when they receive the price offer $p = x - v$ and signal x. This is the situation in which the consumers are indifferent between accepting and rejecting. Note that under this price, they do not mix if the realized signal is strictly lower or higher than x. A special example of η is that $\eta \equiv 1$. This η exactly corresponds to the strategy that consumers always purchase with probability one whenever they are indifferent. Such a triple (σ, η, v) is an equilibrium if (9) (in the article) is satisfied and, for all $b \in \text{supp}(\sigma)$,

$$\begin{aligned} & -(b-v)[(c'_G(b-) - c'_G(b+))\eta(b) + c'_G(b+)] \\ \geq & -(x-v)[(c'_G(x-) - c'_G(x+))\eta(x) + c'_G(x+)], \ \forall x \in [v, 1]. \end{aligned} \tag{F.1}$$

Some explanations are in order. First, consumers are willing to mix in equilibrium only when they are indifferent between purchasing or continuing to search. Hence, such mixture does not change their search incentives. This is why condition (9) does not change. Second, consumers' mixture affects the firms' demand. If a firm sets signal cutoff x (equivalently, charges price $p = x - v$), the demand is $(G(x) - G(x-))\eta(x) + 1 - G(x)$. Because $-c'_G(x-) = 1 - G(x-)$ as before, and because $-c'_G(x+) = 1 - G(x)$, we can express this demand as $-[(c'_G(x-) - c'_G(x+))\eta(x) + c'_G(x+)]$. Therefore, condition (F.1) is simply firms' pricing incentive after accommodating the consumers' mixture. Note that if $\eta \equiv 1$, (F.1) boils down to (10) (in the article). In this case, (σ, η, v) is just the equilibrium (σ, v) in the sense that we considered in Section 5.

The following claim shows that even if $\eta \not\equiv 1$, condition (F.1) still implies (10). This means that if (σ, η, v) is an equilibrium, then (σ, v) must be an equilibrium in the sense that we considered in Section 5. Consequently, it proves that restricting to those equilibria in which the consumers always purchase with probability one when they are indifferent entails no loss of generality.

Claim F.1. *Consider $b \in \mathrm{supp}(\sigma)$. If condition (F.1) is satisfied, so is condition (10).*

Proof. First, we show that if b is an atom of G, then $\eta(b) = 1$. Suppose, by contradiction, $\eta(b) < 1$. When b is an atom, we know $1 - G(x) \geq 1 - G(b-) = (1 - G(b)) + (G(b) - G(b-)) = (1 - G(b)) + \eta(b)(G(b) - G(b-)) + \varepsilon$, where $\varepsilon = (1 - \eta(b))(G(b) - G(b-)) > 0$, for all $x < b$. That is, the demand for cutoff $x < b$ is greater than that for cutoff b by at least $\varepsilon > 0$. When x is sufficiently close to b, the profit from cutoff x must be strictly higher than that from cutoff b. This simply means that (F.1) is violated, a contradiction. Therefore, we must have $\eta(b) = 1$.

The above analysis implies $-(b - v)[(c'_G(b-) - c'_G(b+))\eta(b) + c'_G(b+)] = -(b - v)c'_G(b-)$ if b is an atom. But notice that this equation automatically holds if b is not an atom, as in this case $c'_G(b-) = c'_G(b+)$. Therefore, the left-hand side of (F.1) is simply $-(b - v)c'_G(b-)$, regardless of whether b is an atom or not. But then (F.1) implies $-(b-v)c'_G(b-) \geq -(x-v)c'_G(x-)$ if $x \in [v, 1]$ is not an atom of G. If $x \in [v, 1]$ is an atom, pick a sequence $\{x_n\}$ such that $x_n \uparrow x$ and x_n is not an atom of G for all n. Because $-(b - v)c'_G(b-) \geq -(x_n - v)c'_G(x_n-)$ for all n and $\tilde{x} \mapsto c'_G(\tilde{x}-)$ is left continuous, we know $-(b - v)c'_G(b-) \geq -(x - v)c'_G(x-)$, completing the proof. $\square$

Appendix G Proof of Proposition 6

Proof of Proposition 6. Consider (11) (in the article) as a two-stage optimization problem: choosing v in the first stage and c_1 in the second stage. Given any $v \in [0,\, v^*(s)]$, the second stage optimization is

$$\max_{c_1 \in \mathcal{C}_F} \; c_1(b_1)$$
$$\exists b_1 \text{ s.t. } -(b_1 - v)c_1'(b_1-) \geq -(x - v)c_1'(x-),\; \forall x \in [v, 1].$$

It can be equivalently formulated as

$$\max \; s'$$
$$\exists c_1 \in \mathcal{C}_F \text{ and } b_1 \text{ such that } c_1(b_1) = s', \text{ and}$$
$$-(b_1 - v)c_1'(b_1-) \geq -(x - v)c_1'(x-),\; \forall x \in [v, 1],$$

or equivalently

$$\begin{aligned} &\max \; s' \\ &\text{s.t. } v \leq v^*(s'). \end{aligned} \tag{G.1}$$

Because v^* is strictly decreasing, the solution to (G.1) is obviously $v^{*-1}(v) \in [s, s^*]$, where v^{*-1} is the inverse of v^* function.[9]

Now consider the first stage choice of v. The optimization problem can be written as

$$\max_{v \in [0,\, v^*(s)]} \; v - s + v^{*-1}(v),$$

or equivalently

$$\max_{s' \in [s,\, s^*]} \; v^*(s') - s + s'. \tag{G.2}$$

Applying a similar argument as the proof for Claim C.1, we can show that $v^*(\underline{s}) \geq$

[9]The range of v^* over $[s, s^*]$ is $[v^*(s^*), v^*(s)]$. Because it is possible that $v^*(s^*) > 0$, we define $v^{*-1}(v) = s^*$ if $v \in [0, v^*(s^*))$.

$v^*(\bar{s}) + \frac{\bar{s}-\underline{s}}{\rho^*(\bar{s})}$ for all $\underline{s} < \bar{s}$. This implies that $v^*(s) + s$ is decreasing. Therefore, $s' = s$ is a solution to (G.2), which implies the value of (G.2) and hence the value of (11) is simply $v^*(s)$. Clearly, it can be achieved by choosing $v = v^*$ and c_1 as the symmetric consumer-optimal signal distribution. □